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Novel Non-monotonic Lyapunov-Krasovskii Based Stability Analysis and Stabilization of Discrete State-delay System

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Abstract: This paper proposes a novel less-conservative non-monotonic Lyapunov-Krasovskii stability approach for stability analysis of discrete time-delay systems. In this method, monotonically decreasing requirements of the Lyapunov-Krasovskii method are replaced with non-monotonic ones. The Lyapunov-Krasovskii functional is allowed to increase in some steps, but the overall trend should be decreasing. The model of practical systems used for stability analysis usually contain uncertainty. Therefore, firstly a non-monotonic stability condition is derived for certain discrete time-delay systems, then robust non-monotonic stability conditions are proposed for uncertain systems. Finally, a novel stabilization algorithm is derived based on the introduced non-monotonic stability condition. The Lyapunov-Krasovskii functional and the controller are obtained by solving a set of linear matrix inequalities (LMI) or iterative LMI based nonlinear minimization. The proposed theorems are first evaluated by some numerical examples, and then by simulation and implementation on the pH neutralizing process plant.

Keywords: Lyapunov-Krasovskii functional, discrete state-delay systems, non-monotonic Lyapunov function, robust stability, stabilization.

1 Introduction

One of the most important methods for stability analysis of time-delay systems is the Lyapunov based approach[1, 2], which has been very successful and applicable in control engineering. Nevertheless, the determination of a proper and less conservative Lyapunov functional (LF) for different types of systems is still a serious challenge. This will be more challenging in time-delay systems. Time-delay systems depend not only on the present states but also on the past d steps of the states. Therefore, stability analysis of time-delay systems requires Lyapunov-Krasovskii functional (LKF) rather than Lyapunov function. Many studies have focused on Lyapunov-Krasovskii stability methods to reduce conservatism by some modifications[2]. These modifications are usually completed by adding new summation terms to the equations that arise in the stability analysis procedure. Therefore, a popular conservatism reduction method is estimating a smaller upper bound for common summation terms appearing in the forward difference of a Lyapunov–Krasovskii functional[3]. Considerable progress has been made in finding an LKF for stability analysis and controller design for discrete-time-delay systems. However, in many cases, this common LKF approach has been found to be still very conservative and should be improved more. The free-weighting matrix (FWM) approach[4] and the Jensen-based inequality (JBI) are among the early approaches to reduce the conservatism[5]. This problem was followed by relaxation of derived inequalities in [6, 7]. Also, an improved summation inequality and an Abel lemma-based finite-sum inequality were derived in [8, 9] respectively, by providing tighter bounds of summation term.

In all the aforementioned articles, the LKF and its time derivative include a combination of present and past states of the delay system which are difficult to deal with. Besides, some summation terms will appear in the stability procedure, which are usually replaced by an upper bound. These mentioned problems and monotonically decreasing requirements lead to conservatism. An alternative approach to reduce conservatism is using non-monotonic Lyapunov techniques, in which monotonically decreasing conditions are replaced by non-monotonic one. This way, the m-step difference of LKF should be calculated instead of 1-step. Therefore, the functional is allowed to increase locally, but the overall trend should be decreasing. In this way, LKF is chosen among a larger space of functionals. This technique was first introduced as finite-step Lyapunov method in [10]. The same ap-

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proach was introduced in [11] for stabilization of a class of fuzzy models. The term non-monotonic Lyapunov was first used in [12]. They established global asymptotic stability in discrete time systems by replacing some conditions. Similarly, a discrete non-monotonic Lyapunov method is introduced in [13] by relaxing the monotonicity requirement of Lyapunov′s theorem for stability analysis of fuzzy control systems. The non-monotonic Lyapunov function was developed in [14] for discrete-time switching linear systems. This was followed by presenting a robust H_∞ control for switched systems, which was called N-step ahead Lyapunov function approach[15]. Also, the non-monotonic technique was proposed to design controllers such as optimal controller[16], robust output feedback controller[17], robust state feedback controller[18], and robust H_∞ control for a class of discrete-time nonhomogeneous Markovian jump linear systems[19].

Uncertainty is a major issue in practical problems as it causes the unwanted changes in the system model. Several researchers addressed uncertain discrete time-delay systems in their works. Hui et al.[20] proposed improved delay-dependent robust stability criteria for uncertain systems with interval time-varying delay. The problems of robust stability analysis and robust stabilization of discrete time-delay systems with norm-bounded parameter uncertainties were considered in [21]. Robust stability is studied in [22] for a class of uncertain discrete-time state-delayed systems in state space realization using generalized overflow arithmetic. Then, Shafai et al.[23] focused on a class of non-negative discrete-time-delay systems and showed that the uncertain time-delay systems are asymptotically stable if and only if an associated nonnegative system without delay is asymptotically stable. In [24], the delay-dependent robust stability problem of a class of uncertain discrete-time systems with time-varying delay, and nonlinear perturbations using Lyapunov functional approach was investigated. Finally, the robust preview tracking control and the problem of robust stabilization were proposed in [25, 26] respectively, for uncertain discrete-time-delay systems.

This paper introduces a novel stability analysis theorem for linear discrete time-delay systems to reduce the conservatism, which is called non-monotonic Lyapunov-Krasovskii (NMLK). Due to the Lyapunov function becoming functional in delay systems, mathematical manipulation of LK methods in delay systems cannot easily be applied to the NMLK problem. One of the challenges appears when LKF is calculated for m-step difference and how to deal with the states $x\left(k+j\right),\ \ j=1,\ \cdots,m$, in order to extract the LMI conditions. Some tricks are used in this paper to extract the stability conditions. Also, the robust stability analysis for the same problem is provided using NMLK-based stability criteria. Due to the model changes caused by uncertainty, the LKF can be incremental in some steps which leads to instability. This is one of the problems that our proposed technique can overcome, because the robust NMLK does not need strictly decreasing LKF. In the robust version of NMLK, novel lemmas are introduced and used in the proposed innovative procedure for deriving LMI criteria. Finally, stabilization criteria are derived based on the NMLK theorem. The stabilization conditions are achieved through targeted manipulation which are explained throughout the proof. In this regard, a controller is designed for a pH neutralizing process plant and applied experimentally on a laboratory pilot plant using the proposed stabilization theorem. This practical implementation helps to investigate the efficacy of the derived stabilization algorithm. The NMLK-based stability analysis and controller design usually lead to an optimization or linear matrix inequality (LMI) based nonlinear minimization problems, which can be solved either using LMIs approaches.

The rest of this paper continues by reviewing some definitions and required preliminaries in Section 2. Then, Section 3 provides the main results of the study, which are the NMLK stability and robust stability conditions. The stabilization criterion is derived based on NMLK method in Section 3.3. Numerical examples evaluate the efficacy of the proposed algorithms in Section 4. Section 5 provides some simulation and experimental results. A discussion of the innovations and novelty of the paper is provided in Section 6. The paper is concluded in Section 7.

Notations. Throughout this paper, for matrices X and Y, the notation $X\geq Y$ (respectively, $X>Y$) means that the matrix $X-Y$ is positive semi-definite (respectively, positive definite). Matrices are assumed to have compatible dimensions if their dimensions are not explicitly stated. $\lambda_{\max}\left(A\right)$ is the maximum eigenvalue of the matrix A. I_n is the identity matrix. For simplicity, we show the spectral norm $\left\|A\right\|_2=\sqrt{\lambda_{\max}\left(A^{\rm T}A\right)}$ with $\|A\|$ throughout the paper. The notation * represents symmetric terms in a symmetric matrix.

2 Preliminaries and definitions

In this article, the objective is to reduce the conservatism of Lyapunov-Krasovskii (LK) stability analysis of discrete-time state-delay (DTSD) systems using a non-monotonic decreasing technique. LK stability theorems are extensively used to study the stability of delayed systems. In the stability analysis using NMLK functional, it is not necessary for the functional to decrease monotonically. Instead, it is sufficient to prove that its trend is decreasing, while it allows increasing locally for a few steps. Let $\forall\theta\in[-d,\ 0]:\ \ x\left(k+\theta\right)\in{\rm C}$ which ${\rm C}:[-d,\ 0]\to{\bf R}^n$ is the space of functions with supremum norm. Let $\bar{x}=\left(x\left(k\right)^{\rm T},\ \ x(k-d)^{\rm T}\right)^{\rm T}$. Consider the DTSD system (1) as follows:

$$\begin{cases}x\left(k+1\right)=Ax\left(k\right)+A_dx\left(k-d\right)\\ x\left(\theta\right)=\varPhi\left(\theta\right),\theta=-d,-d+1,\cdots,0\end{cases}\qquad(1)$$

where A and A_d are system coefficient matrices with appropriate dimensions and $\varPhi\left(\theta\right)$ are the initial conditions. Then, the standard monotonic LK theorem was presented as follows.

Theorem 1. The solution $x = 0$ of the system (1) is asymptotically stable if there exist continues LKF $V : \mathbf{R}^n \to \mathbf{R}$ and continues non-decreasing functions v and $w : \mathbf{R}^+ \to \mathbf{R}^+$ with feature $v(0) = w(0) = 0$, $v(s) > 0$ and $w > 0\ \forall\ s > 0$, such that $\forall x_k \in \mathbf{R}^n$:

1) $0 < V(x_k) \le v(\|\bar{x}\|)$

2) $V(0) = 0$

3) $V(x_k) \cong V(x_{k+1}) - V(x_k) \le -w(\|\bar{x}\|)$.

Lemma 1 is used for obtaining the new non-monotonic based stability criterion.

Lemma 1.[9] For a constant matrix $Z \in \mathbf{R}^{n\times n}$ with $Z = Z^{\mathrm{T}} > 0$, integers r_1 and r_2 with $r_2 - r_1 > 1$, the following inequality holds:

$$\sum_{j=r_1}^{r_2-1} \eta^{\mathrm{T}}(j) Z\eta(j) \ge \frac{1}{\ell_1}\upsilon_1^{\mathrm{T}} Z\upsilon_1 + \frac{3\ell_2}{\ell_1\ell_3}\upsilon_2^{\mathrm{T}} Z\upsilon_2 \tag{2}$$

where $\ell_1 = r_2 - r_1$, $\ell_2 = r_2 - r_1 - 1$, $\ell_3 = r_2 - r_1 + 1$, $\eta(j) = x(j+1) - x(j)$, $\upsilon_1 = x(r_2) - x(r_1)$ and $\upsilon_2 = x(r_2) + x(r_1) - \frac{2}{\ell_2}\sum_{j=r_1+1}^{r_2-1} x(j)$.

The main results of this study is provided in Section 3.

3 Main results

In the all of the available LK stability approaches, the LK functional must be strictly decreasing. In this section, the new non-monotonic Lyapunov-Krasovskii stability criteria is introduced. In the NMLK theorem, m-step forward difference is defined as (3):

$$\Delta_m V(x_k) \triangleq V(x_{k+m}) - V(x_k). \tag{3}$$

The NMLK functional is permitted to increase locally within every m-step, but it must be decreasing in general trend. The parameter m is called non-monotonicity step. The state space representation of DTSD system is assumed as system (1). In this regard, first the novel NMLK theorem is proposed for stability analysis of DTSD systems. Then, the robust stability criteria for time-delay systems is introduced in Section 3.2. Finally, a stabilization criterion will be provided in Section 3.3 based on NMLK technique.

3.1 Non-monotonic Lyapunov-Krasovskii stability analysis method

In the non-monotonic stability approach, NMLK functional can be up to m-steps incremental but the overall trend must be decreasing. Using the forward difference operator $\Delta_m V(x_k)$, Theorem 2 is obtained which gives the sufficient conditions for delay-dependent stability of DTSD system (1). Consider the NMLK functional candidate in the form of $V = V_1 + V_2 + V_3$, which V_1, V_2 and V_3 are expressed in (4a)–(4c):

$$V_1(k) = X^{\mathrm{T}}(k) PX(k) \tag{4a}$$

$$V_2(k) = \sum_{j=k-d}^{k-1} x^{\mathrm{T}}(j) Qx(j) \tag{4b}$$

$$V_3(k) = \sum_{\theta=-d}^{-1}\sum_{j=k+\theta}^{k-1} \eta^{\mathrm{T}}(j) Z\eta(j) \tag{4c}$$

where $X = \begin{bmatrix} x(k) \\ \sum_{j=k-d}^{k-1} x(j) \end{bmatrix}$ and $(j) = x(j+1) - x(j)$. Also, let us define the vectors $\xi^{\mathrm{T}}(x)$ and e_i as

$$\xi^{\mathrm{T}}(x) = [x(k), x(k-d), x(k-2d), \cdots, x(k-md), \sum_{j=k-d+m-1}^{k-1} x^{\mathrm{T}}(j)] \tag{5a}$$

$$e_i = \begin{bmatrix} 0_{n\times(i-1)n} & I_{n\times n} & 0_{n\times(m+2-i)n} \end{bmatrix}, \quad i = 1, \cdots, (m+2) \tag{5b}$$

The selection of $\xi^{\mathrm{T}}(x)$ should be such that simultaneously have suitable form for both stability analysis calculations and stabilization condition extraction in Section 3.3. More information can be found in Section 3.3, followed by some discussion in Section 6.

Theorem 2. Linear DTSD system (1) is m-step non-monotonic Lyapunov-Krasovskii stable, with $1 \le m < d$, if there exist positive definite matrices $P \in \mathbf{R}^{2n\times 2n}$, $Q \in \mathbf{R}^{n\times n}$ and $Z \in \mathbf{R}^{n\times n}$ such that:

$$\varPsi_1 + \varPsi_2 + \varPsi_3 < 0 \tag{6}$$

where

$$\varPsi_1 = \varXi_1^{\mathrm{T}} P\varXi_1 - \varXi_2^{\mathrm{T}} P\varXi_2 \tag{7}$$

$$\varPsi_2 = \sum_{j=0}^{m-1}\left(\left(L_j^1\right)^{\mathrm{T}} QL_j^1 - \left(L_j^2\right)^{\mathrm{T}} QL_j^2\right) \tag{8}$$

$$\begin{aligned}\varPsi_3 = &\sum_{j=0}^{m-1}(d-m+j+1)\left(L_{j+1}^1 - L_j^1\right)^{\mathrm{T}} Z\left(L_{j+1}^1 - L_j^1\right) - \\ &\sum_{j=0}^{m-2}(j+1)\left(L_{j+1}^2 - L_j^2\right)^{\mathrm{T}} Z\left(L_{j+1}^2 - L_j^2\right) - \\ &\frac{m}{\ell_1}(\varphi_m)^{\mathrm{T}} Z(\varphi_m) - \frac{3m\ell_2}{\ell_1\ell_3}(\varPi_m)^{\mathrm{T}} Z(\varPi_m)\end{aligned} \tag{9}$$

$$\varXi_1 = \begin{bmatrix} L_m^1 \\ e_{m+2} - L_{m-1}^2 + \sum_{j=0}^{m-1} L_j^1 \end{bmatrix},$$

$$\varXi_2 = \begin{bmatrix} e_1 \\ e_{m+2} + \sum_{j=0}^{m-2} L_j^2 \end{bmatrix} \tag{10a}$$

$$\varPi_m = e_1 + \frac{\ell_2 + 2}{\ell_2} L_{m-1}^2 - \frac{2}{\ell_1} e_{m+2} \tag{10b}$$

$$\varphi_m = e_1 - L_{m-1}^2 \tag{10c}$$

$$\ell_1 = d - m + 1, \ell_2 = d - m, \ell_3 = d - m + 2 \tag{10d}$$

$$\begin{aligned} &\begin{cases} L_0^1 = e_1 \\ L_j^1 = AL_{j-1}^1 + A_d L_{j-1+(1)}^1 \end{cases} (j = 1, \cdots, m) \\ &\begin{cases} L_0^2 = e_2 \\ L_j^2 = AL_{j-1}^2 + A_d L_{j-1+(1)}^2 \end{cases} (j = 1, \cdots, (m-1)) \end{aligned} \tag{10e}$$

in which $L_{j-1+(1)}^1$ and $L_{j-1+(1)}^2$ will be defined later in Remark 2.

Proof. To analyze the stability of the system (1), $\Delta_m V$ should be extracted. In this regard, $\Delta_m V_1$, $\Delta_m V_2$ and $\Delta_m V_3$ can be extracted separately using (4a)–(4c), and finally $\Delta_m V = \Delta_m V_1 + \Delta_m V_2 + \Delta_m V_3 < 0$ yields the stability criteria.

1) **Calculation of $\Delta_m V_1$**

Using (4a), the m-step forward difference is formulated as the following equation:

$$\Delta_m V_1 = X^{\rm T}(k+m) PX(k+m) - X^{\rm T}(k) PX(k) \tag{11}$$

which can be written as

$$\begin{aligned} \Delta_m V_1 = &\begin{bmatrix} x(k+m) \\ \sum_{j=k-d}^{k-1} x(j+m) \end{bmatrix}^{\rm T} P \begin{bmatrix} x(k+m) \\ \sum_{j=k-d}^{k-1} x(j+m) \end{bmatrix} - \\ &\begin{bmatrix} x(k) \\ \sum_{j=k-d}^{k-1} x(j) \end{bmatrix}^{\rm T} P \begin{bmatrix} x(k) \\ \sum_{j=k-d}^{k-1} x(j) \end{bmatrix}. \end{aligned} \tag{12}$$

Using the state space representation (1), the m-step ahead state $x(k+m)$ is expanded as follows (see Appendix B):

$$x(k+m) = L_m^1 \xi(x) \tag{13}$$

where L_m^1 is defined in (10e). On the other hand, by expanding the summation terms $\sum_{j=k-d}^{k-1} x(j+m)$ and $\sum_{j=k-d}^{k-1} x(j)$ in (12), we have

$$\begin{aligned} \sum_{j=k-d}^{k-1} x(j+m) = &\sum_{j=k-d+m-1}^{k-1} x(j) - \\ &x(k-d+m-1) + \sum_{j=0}^{m-1} x(k+j) \end{aligned} \tag{14a}$$

$$\sum_{j=k-d}^{k-1} x(j) = \left(\sum_{j=k-d+m-1}^{k-1} x(j) \right) + \sum_{j=-d}^{-d+m-2} x(k+j). \tag{14b}$$

Using (5a), we have $\sum_{j=k-d+m-1}^{k-1} x(j) = e_{m+2} \xi(x)$.

Also, according to Appendix B, j-step ahead states can be rewritten in the form of $L_j^i s$. Then, (15a) and (15b) are obtained by some manipulations:

$$\sum_{j=k-d}^{k-1} x(j+m) = \left(e_{m+2} - L_{m-1}^2 + \sum_{j=0}^{m-1} L_j^1 \right) \xi(x) \tag{15a}$$

$$\sum_{j=k-d}^{k-1} x(j) = \left(e_{m+2} + \sum_{j=0}^{m-2} L_j^2 \right) \xi(x). \tag{15b}$$

Then, using (13)–(15), equation (16) is reached:

$$\begin{aligned} \Delta_m V_1 = \xi^{\rm T}(x) \Bigg(&\begin{bmatrix} L_m^1 \\ e_{m+2} - L_{m-1}^2 + \sum_{j=0}^{m-1} L_j^1 \end{bmatrix}^{\rm T} \times \\ &P \begin{bmatrix} L_m^1 \\ e_{m+2} - L_{m-1}^2 + \sum_{j=0}^{m-1} L_j^1 \end{bmatrix} - \\ &\begin{bmatrix} e_1 \\ e_{m+2} + \sum_{j=0}^{m-2} L_j^2 \end{bmatrix}^{\rm T} \\ &P \begin{bmatrix} e_1 \\ e_{m+2} + \sum_{j=0}^{m-2} L_j^2 \end{bmatrix} \Bigg) \xi(x). \end{aligned} \tag{16}$$

2) **Calculation of $\Delta_m V_2$**

$\Delta_m V_2$ is calculated using the m-step forward difference of (4b). Therefore,

$$\begin{aligned} \Delta_m V_2 = &\sum_{j=k-d}^{k-1} x^{\rm T}(j+m) Qx(j+m) - \\ &\sum_{j=k-d}^{k-1} x^{\rm T}(j) Qx(j). \end{aligned} \tag{17}$$

By expanding both sigma terms in (17) and removing similar terms with opposite signs, we get

$$\begin{aligned} &\Delta_m V_2 = x^{\rm T}(k+m-1) Qx(k+m-1) + \cdots + \\ &x^{\rm T}(k+1) Qx(k+1) + x^{\rm T}(k) Qx(k) - \\ &x^{\rm T}(k-d+m-1) Qx(k-d+m-1) - \cdots - \\ &x^{\rm T}(k-d+1) Qx(k-d+1) - x^{\rm T}(k-d) Qx(k-d) \end{aligned} \tag{18}$$

In (18), j-step ahead states are calculated along with the solution of system (1). Using (5a) and representing each term in (18) in the form of L_j^1 and L_j^2 (see Appendix B):

$$\Delta_m V_2 = \xi^{\mathrm{T}}(x)\Big\{\left(L_{m-1}^1\right)^{\mathrm{T}} Q L_{m-1}^1 + \cdots + \left(L_1^1\right)^{\mathrm{T}} Q L_1^1 + \left(L_0^1\right)^{\mathrm{T}} Q L_0^1 - \left(L_{m-1}^2\right)^{\mathrm{T}} Q L_{m-1}^2 - \cdots - \left(L_1^2\right)^{\mathrm{T}} Q L_1^2 - \left(L_0^2\right)^{\mathrm{T}} Q L_0^2\Big\}\xi(x). \qquad (19)$$

Then, simplifying the representation yields

$$\Delta_m V_2 = \xi^{\mathrm{T}}(x)\left\{\sum_{j=0}^{m-1}\left(\left(L_j^1\right)^{\mathrm{T}} Q L_j^1 - \left(L_j^2\right)^{\mathrm{T}} Q L_j^2\right)\right\}\xi(x) \qquad (20)$$

3) **Calculation of ΔV_3**

The m-step forward difference $\Delta_m V_3$ is calculated as

$$\Delta_m V_3 = \sum_{\theta=-d}^{-1}\sum_{j=k+\theta}^{k-1}\Big(\eta^{\mathrm{T}}(j+m) Z \eta(j+m) - \eta^{\mathrm{T}}(j) Z \eta(j)\Big). \qquad (21)$$

By expanding the summations, (21) can be represented as

$$\Delta_m V_3 = d\eta^{\mathrm{T}}(k) Z \eta(k) + \cdots + d\eta^{\mathrm{T}}(k+m-1) Z \eta(k+m-1) - \sum_{j=k-d}^{k-1}\eta^{\mathrm{T}}(j) Z \eta(j) - \sum_{j=k-d+1}^{k}\eta^{\mathrm{T}}(j) Z \eta(j) - \cdots - \sum_{j=k-d+m-1}^{k+m-2}\eta^{\mathrm{T}}(j) Z \eta(j). \qquad (22)$$

Using (5a) and the result of Appendix B, $\Delta_m V_3$ is simplified as follows:

$$\Delta_m V_3 = \xi^{\mathrm{T}}(x)\Bigg(\sum_{j=0}^{m-1}(d-m+1+j)\left(L_{j+1}^1 - L_j^1\right)^{\mathrm{T}} \times Z\left(L_{j+1}^1 - L_j^1\right) - \sum_{j=0}^{m-2}(j+1)\left(L_{j+1}^2 - L_j^2\right)^{\mathrm{T}} \times Z\left(L_{j+1}^2 - L_j^2\right)\Bigg)\xi(x) + \Gamma \qquad (23)$$

where $\Gamma = -m\sum_{j=k-d+m-1}^{k-1}\eta^{\mathrm{T}}(j) Z \eta(j)$.

The term Γ in (23) is negative definite and can be omitted in the stability analysis. But, instead of removing it, we try to calculate an upper bound for it, which helps to lessen the conservatism. For this purpose, considering $\eta(j) = x(j+1) - x(j)$ and using Lemma 1 (Abel Lemma), we have

$$-m\sum_{j=k-d+m-1}^{k-1}\eta^{\mathrm{T}}(j) Z \eta(j) \le -\frac{m}{\ell_1}(x(k) - x(k-d+m-1))^{\mathrm{T}} Z (x(k) - x(k-d+m-1)) - \frac{3m\ell_2}{\ell_1\ell_3}\left(x(k) + x(k-d+m-1) - \frac{2}{\ell_2}(-x(k-d+m-1) + \sum_{j=k-d+m-1}^{k-1} x(j))\right)^{\mathrm{T}} Z\left(x(k) + x(k-d+m-1) - \frac{2}{\ell_2}\left(-x(k-d+m-1) + \sum_{j=k-d+m-1}^{k-1} x(j)\right)\right). \qquad (24)$$

Therefore, $\Delta_m V_3$ can be represented in the form of (25) using the calculated upper bound.

$$\Delta_m V_3 = \xi^{\mathrm{T}}(x)\Bigg(\sum_{j=0}^{m-1}(d-m+1+j)\left(L_{j+1}^1 - L_j^1\right)^{\mathrm{T}} Z\left(L_{j+1}^1 - L_j^1\right) - \sum_{j=0}^{m-2}(j+1)\left(L_{j+1}^2 - L_j^2\right)^{\mathrm{T}} Z\left(L_{j+1}^2 - L_j^2\right) - \frac{m}{\ell_1}(\varphi_m)^{\mathrm{T}} Z(\varphi_m) - \frac{3m\ell_2}{\ell_1\ell_3}(\Pi_m)^{\mathrm{T}} Z(\Pi_m)\Bigg)\xi(x) \qquad (25)$$

where ϕ_m, Π_m, ℓ_1, ℓ_2 and ℓ_3 are defined in (10b)–(10f). $\Delta_m V$ consists of $\Delta_m V_1$, $\Delta_m V_2$ and $\Delta_m V_3$ which were calculated in the previous parts. As we know, the system (1) is stable if $\Delta_m V = \Delta_m V_1 + \Delta_m V_2 + \Delta_m V_3 < 0$.

Replacing $\Delta_m V_1$, $\Delta_m V_2$ and $\Delta_m V_3$ from (16), (20) and (25) and after some manipulations, the stability sufficient conditions are obtained as stated in (6). □

Remark 1. Commonly, discrete Jensen's inequality lemma is used in the stability analysis of time-delay systems. Abel lemma is used in this paper instead of Jensen's inequality lemma to reduce the conservatism by providing a tighter lower bound[9].

Remark 2. L_j^1 and L_j^2 are calculated using Algorithm 1.

Algorithm 1. Calculating $L_j^l, (l = 1, 2)$

If $l = 1$ then $j = 1, \cdots, m$
If $l = 2$ then $j = 1, \cdots, m-1$

Step 1 : Let $\to L_0^l = e_l$ and $k = 0$

Step 2 : $\begin{cases}\text{Find } L_{k+(1)}^l \text{ by shifting column} \\ e_i s \text{ in } L_k^l \text{ one step to the right}\end{cases}$

⋮

Step 3 : $\begin{cases}\text{Let} \to L_{k+1}^l = A L_k^l + A_d L_{k+(1)}^l \\ \text{and } k = k+1\end{cases}$

⋮

Step 4 : Continue Steps 2 and 3 until $k = j$

In Algorithm 1, $L_{k+(1)}^1$ and $L_{k+(1)}^2$ mean that the indexes of e_is in L_k^l are shifted one step to the right. For instance, the first three steps of Algorithm 1 are represented in the following:

1) $L_0^1 = e_1$
2) $L_1^1 = AL_0^1 + A_d L_{0+(1)}^1 = Ae_1 + A_d e_{1+1} = Ae_1 + A_d e_2$
3) $L_2^1 = AL_1^1 + A_d L_{1+(1)}^1 = A(Ae_1 + A_d e_2) + A_d(Ae_{1+1} + A_d e_{2+1}) = A^2 e_1 + AA_d e_2 + A_d A e_2 + A_d^2 e_3.$

Remark 3. Non-monotonicity step in Theorem 2 can reduce the conservatism in the cost of increasing the calculations. Therefore, choosing a proper value for the parameter m is a trade-off. As m increases, the conservativeness decreases but the calculation will increase. Our experience shows that even a 2-step of non-monotonicity significantly decreases the conservatism, while has less computations compared with larger value for the m-step. This means, in many applications, it might be sufficient to have 2-step non-monotonicity. In this regard, Corollary 1 renders the 2-step NMLK stability condition of the system (1).

First, let e_i $(i = 1, 2, 3, 4)$ be $n \times 4n$ block-row vectors of the $n \times n$ identity matrix so that $e_i = [0_{n\times(i-1)n} \quad I_{n\times n} \quad 0_{n\times(4-i)n}]$.

Corollary 1. Linear DTSD system (1) with the given initial condition is 2-step non-monotonic Lyapunov-Krasovskii stable if there exist positive definite matrices $P \in \mathbf{R}^{2n\times 2n}$, $Q \in \mathbf{R}^{n\times n}$ and $Z \in \mathbf{R}^{n\times n}$ such that

$$\Psi_1 + \Psi_2 + \Psi_3 < 0 \tag{26}$$

where

$$\Psi_1 = \Xi_1^{\rm T} P \Xi_1 - \Xi_2^{\rm T} P \Xi_2 \tag{27}$$

$$\Psi_2 = \left(L_0^1\right)^{\rm T} Q L_0^1 + \left(L_1^1\right)^{\rm T} Q L_1^1 - \left(L_0^2\right)^{\rm T} Q L_0^2 - \left(L_1^2\right)^{\rm T} Q L_1^2 \tag{28}$$

$$\begin{aligned}\Psi_3 = &\ (d-1)\left(L_1^1 - L_0^1\right)^{\rm T} Z\left(L_1^1 - L_0^1\right) + \\ & d\left(L_2^1 - L_1^1\right)^{\rm T} Z\left(L_2^1 - L_1^1\right) - \left(L_1^2 - L_0^2\right)^{\rm T} Z\left(L_1^2 - L_0^2\right) - \\ & \frac{2}{d-1}\varphi_2{}^{\rm T} Z \varphi_2 - \frac{6(d-2)}{(d)(d-1)}\varPi_2^{\rm T} Z \varPi_2\end{aligned} \tag{29}$$

$$\Xi_1 = \begin{bmatrix} L_2^1 \\ L_0^1 + L_1^1 - L_1^2 + e_4 \end{bmatrix}, \Xi_2 = \begin{bmatrix} e_1 \\ L_0^2 + e_4 \end{bmatrix} \tag{30a}$$

$$\begin{aligned} & L_0^1 = e_1, L_1^1 = Ae_1 + A_d e_2 \\ & L_2^1 = A^2 e_1 + (AA_d + A_d A)e_2 + A_d^2 e_3 \\ & L_0^2 = e_2 \\ & L_1^2 = Ae_2 + A_d e_3 \\ & \varphi_2 = \left(L_0^1 - L_1^2\right) \end{aligned} \tag{30b}$$

$$\varPi_2 = L_0^1 + \frac{d}{(d-2)}L_1^2 - \frac{2}{(d-2)}e_4. \tag{30c}$$

3.2 Robust non-monotonic Lyapunov-Krasovskii stability analysis method

Almost all the practical systems that are modeled in the terms of mathematical equations are subject to uncertainty. Therefore, providing a stability analysis approach that takes the uncertainty into account is an important step of stability analysis in any control system. In this section, we give the results on the robust non-monotonic Lyapunov-Krasovskii (RNMLK) stability analysis for the uncertain time-delay system (31):

$$\begin{cases} x(k+1) = (A + \delta A(k))\, x(k) + \\ \qquad\qquad (A_d + \delta A_d(k))\, x(k-d) \\ x(\theta) = \varPhi(\theta), \theta = -d, -d+1, \cdots, 0 \end{cases} \tag{31}$$

where $\varPhi(\theta)$ are the initial conditions, $\delta A(k)$ and $\delta A_d(k)$ are uncertainty matrices in the form of:

$$\begin{bmatrix} \delta A(k) & \delta A_d(k) \end{bmatrix} = MF(k)\begin{bmatrix} N_A & N_d \end{bmatrix} \tag{32}$$

in which N_A, N_d and M are constant matrices and $F(k)$ is the uncertainty matrix that satisfies the following inequality:

$$F(k)^{\rm T} F(k) \le I. \tag{33}$$

Theorem 3 implies the RNMLK stability theorem with the non-monotonicity step $m = 2$.

Theorem 3. Linear discrete uncertain time-delay system (31) is 2-step RNMLK stable if there exist positive definite matrices $P \in \mathbf{R}^{2n\times 2n}$, $Q \in \mathbf{R}^{n\times n}$, $Z \in \mathbf{R}^{n\times n}$ and $\varepsilon_i > 0$, $i = 1, \cdots, 5$ such that:

$$\Psi_{11} + \begin{bmatrix} 0_{4n\times 4n} & [\Psi_{12}^{\rm T}] \\ [\Psi_{12}] & \Psi_{22} \end{bmatrix} < 0 \tag{34}$$

where

$$\begin{aligned}\Psi_{12}^{\rm T} = \Big[& \varphi_2^{\rm T} Z M, \quad \varPi_2^{\rm T} Z M, \quad \left(L_1^2\right)^{\rm T} Q M, \ \ldots \\ & \left(L_1^2 - L_0^2\right)^{\rm T} Z M, \quad \left(L_1^1\right)^{\rm T}, \ \sqrt{d-1}\left(\bar{L}_2 - L_1^1\right)^{\rm T}, \cdots, \\ & \sqrt{d}\left(L_1^1 - L_0^1\right)^{\rm T}, \quad \Xi_1^{\rm T} \Big]\end{aligned}$$

$$\Psi_{22}=\left[\begin{array}{cccccc} 0_{4n\times 4n} & 0_{4n\times n} & 0_{4n\times n} & 0_{4n\times n} & \left[\begin{array}{cc} 0_{4n\times n} & 0_{4n\times n}\end{array}\right] \\ 0_{n\times 4n} & \lambda & -\sqrt{d-1}\lambda & \sqrt{d}\lambda & \left[\begin{array}{cc} 0_{n\times n} & \lambda\end{array}\right] \\ 0_{n\times 4n} & -\sqrt{d-1}\lambda & (d-1)\lambda & -\sqrt{d(d-1)}\lambda & \left[\begin{array}{cc} 0_{n\times n} & -\sqrt{d-1}\lambda\end{array}\right] \\ 0_{n\times 4n} & \sqrt{d}\lambda & -\sqrt{d(d-1)}\lambda & d\lambda & \left[\begin{array}{cc} 0_{n\times n} & \sqrt{d}\lambda\end{array}\right] \\ \left[\begin{array}{c} 0_{n\times 4n} \\ 0_{n\times 4n}\end{array}\right] & \left[\begin{array}{c} 0_{n\times n} \\ \lambda\end{array}\right] & \left[\begin{array}{c} 0_{n\times n} \\ -\sqrt{d-1}\lambda\end{array}\right] & \left[\begin{array}{c} 0_{n\times n} \\ \sqrt{d}\lambda\end{array}\right] & \left[\begin{array}{cc} 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & \lambda\end{array}\right] \end{array}\right] \tag{35a}$$

$$\begin{aligned}\Psi_{11}=\mathrm{diag}\{&(\omega+\bar{\omega}+\varepsilon_5^{-1}(N_Ae_1+N_de_2)^{\mathrm{T}}(N_Ae_1+\\&N_de_2)),-\frac{2}{(d-1)}(M^{\mathrm{T}}ZM-\varepsilon_1),-\frac{6(d-2)}{d(d-1)}\times\\&(M^{\mathrm{T}}ZM-\varepsilon_2),-(M^{\mathrm{T}}QM-\varepsilon_3),-(M^{\mathrm{T}}ZM-\\&\varepsilon_4),-Q^{-1},-Z^{-1},-Z^{-1},-P^{-1}\}\end{aligned} \tag{35b}$$

$$\begin{aligned}\bar{\omega}=&-\left(\frac{2}{(d-1)}\varepsilon_1+\frac{6d}{(d-2)(d-1)}\varepsilon_2+\varepsilon_3+\varepsilon_4\right)J_1^{\mathrm{T}}J_1-\\&\frac{2}{(d-1)}\varphi_2^{\mathrm{T}}Z\varphi_2-\frac{6(d-2)}{d(d-1)}\Pi_2^{\mathrm{T}}Z\Pi_2-\left(L_1^2\right)^{\mathrm{T}}QL_1^2-\\&\left(L_1^2-L_0^2\right)^{\mathrm{T}}Z\left(L_1^2-L_0^2\right)\end{aligned} \tag{35c}$$

$$\omega=\left(L_0^1\right)^{\mathrm{T}}QL_0^1-\varXi_2^{\mathrm{T}}P\varXi_2-\left(L_0^2\right)^{\mathrm{T}}QL_0^2 \tag{35d}$$

$$\begin{aligned}&\bar{\varXi}_1=\left[\begin{array}{c}\bar{L}_2^1\\ L_0^1+L_1^1+\bar{L}_1^2+e_4\end{array}\right],\ \varXi_2=\left[\begin{array}{c}e_1\\ (L_0^2+e_4)\end{array}\right]\\&L_0^1=e_1, L_1^1=Ae_1+A_de_2\\&L_0^2=e_2, L_1^2=Ae_2+A_de_3\end{aligned} \tag{35e}$$

$$\begin{aligned}\bar{L}_2^1=&\left(||A||^2+2||A||||M||||N_A||+||M||^2||N_A||^2\right)e_1+\\&2\left(||AA_d||+||A||||M||||N_d||+||M||||N_A||||A_d||+\right.\\&\left.||M||||N_A||||M||||N_d||\right)e_2+\\&\left(||A_d||^2+2||A_d||||M||||N_d||+||M||^2||N_d||^2\right)e_3\end{aligned} \tag{35f}$$

$$\begin{aligned}\bar{L}_1^2=&(||A||+||A||||M||||N_d||+||M||||N_A||||A_d||+\\&||M||||N_A||||M||||N_d||)e_2+\\&\left(||A_d||^2+2||A_d||||M||||N_d||+||M||^2||N_d||^2\right)e_3\end{aligned} \tag{35g}$$

$$\begin{aligned}&\varphi_2=\left(L_0^1-L_1^2\right),\Pi_2=L_0^1+\frac{d}{(d-2)}L_1^2-\frac{2}{d-2}e_4\\&J_1=N_Ae_2+N_de_3,\ \ \lambda=\varepsilon_5MM^{\mathrm{T}}.\end{aligned} \tag{35h}$$

Proof. Considering the uncertain coefficient matrices $\tilde{A}=A+\delta A(k)$, $\tilde{A}_d=A_d+\delta A_d(k)$ and taking the same procedure as the proof of Theorem 2, the 2-step stability condition of the uncertain time-delay system (31) is given as follows:

$$\tilde{\varPsi}_1+\tilde{\varPsi}_2+\tilde{\varPsi}_3<0 \tag{36}$$

where

$$\tilde{\varPsi}_1=\tilde{\varXi}_1^{\mathrm{T}}P\tilde{\varXi}_1-\varXi_2^{\mathrm{T}}P\varXi_2 \tag{37a}$$

$$\tilde{\varPsi}_2=\left(\tilde{L}_1^1\right)^{\mathrm{T}}Q\tilde{L}_1^1+\left(L_0^1\right)^{\mathrm{T}}QL_0^1-\left(L_0^2\right)^{\mathrm{T}}QL_0^2-\left(\tilde{L}_1^2\right)^{\mathrm{T}}Q\tilde{L}_1^2 \tag{37b}$$

$$\begin{aligned}\tilde{\varPsi}_3=&(d-1)\left(\tilde{L}_1^1-L_0^1\right)^{\mathrm{T}}Z\left(\tilde{L}_1^1-L_0^1\right)+d\left(\tilde{L}_2^1\tilde{L}_1^1\right)^{\mathrm{T}}-\\&Z\left(\tilde{L}_2^1-\tilde{L}_1^1\right)-\left(\tilde{L}_1^2-L_0^2\right)^{\mathrm{T}}Z\left(\tilde{L}_1^2-L_0^2\right)-\\&\frac{2}{d-1}\tilde{\varphi}_2^{\mathrm{T}}Z\tilde{\varphi}_2-\frac{6(d-2)}{d(d-1)}\tilde{\Pi}_2^{\mathrm{T}}Z\tilde{\Pi}_2\end{aligned} \tag{37c}$$

and

$$\begin{aligned}&\tilde{\varXi}_1=\left[\begin{array}{c}\tilde{L}_2^1\\ L_0^1+\tilde{L}_1^1-\tilde{L}_1^2+e_4\end{array}\right],\ \varXi_2=\left[\begin{array}{c}e_1\\ (L_0^2+e_4)\end{array}\right],\\&\tilde{\varphi}_2=\left(L_0^1-\tilde{L}_1^2\right),\ \ \tilde{\Pi}_2=L_0^1+\frac{d}{(d-2)}\tilde{L}_1^2-\frac{2}{(d-2)}e_4\\&L_0^1=e_1,\ \tilde{L}_1^1=\tilde{A}e_1+\tilde{A}_de_2,\ \ \tilde{L}_1^2=\tilde{A}e_2+\tilde{A}_de_3\\&\tilde{L}_2^1=\tilde{A}^2e_1+\left(\tilde{A}\tilde{A}_d+\tilde{A}_d\tilde{A}\right)e_2+\tilde{A}_d^2e_3,\ \ L_0^2=e_2.\end{aligned}$$

In the above conditions, the parameters with a tilde (~) contain uncertain matrices $\tilde{A}$ or $\tilde{A}_d$. So, the uncertain parts are $\tilde{\varXi}_1^{\mathrm{T}}P\tilde{\varXi}_1$, defined parameter $\delta\omega$ which is $\delta\omega=-\left(\tilde{L}_1^2\right)^{\mathrm{T}}Q\tilde{L}_1^2-\left(\tilde{L}_1^2-L_0^2\right)^{\mathrm{T}}Z\left(\tilde{L}_1^2-L_0^2\right)-\frac{2}{d-1}\tilde{\varphi}_2^{\mathrm{T}}Z\tilde{\varphi}_2-\frac{6(d-2)}{d(d-1)}\tilde{\Pi}_2^{\mathrm{T}}Z\tilde{\Pi}_2$ and the other parameters containing $\tilde{L}_1^2$, $\tilde{L}_2^1$, $\tilde{L}_1^1$. In order to extract the robust stability conditions, first, an upper bound will be calculated for the uncertain parts $\tilde{\varXi}_1^{\mathrm{T}}P\tilde{\varXi}_1$, $\tilde{L}_1^2$, $\tilde{L}_2^1$ and $\delta\omega$ according to Appendix A. Therefore, using (A15), (A16), (A20) and (A32) in Appendix A, inequality (36) can be written as follows:

$$\begin{aligned}\tilde{\varPsi}_1+\tilde{\varPsi}_2+\tilde{\varPsi}_3<&(d-1)\left(\tilde{L}_1^1-L_0^1\right)^{\mathrm{T}}Z\left(\tilde{L}_1^1-L_0^1\right)+\omega+\\&\bar{\omega}+\bar{\varXi}_1^{\mathrm{T}}P\bar{\varXi}_1+\left(\tilde{L}_1^1\right)^{\mathrm{T}}Q\tilde{L}_1^1+d\left(\bar{L}_2^1-\tilde{L}_1^1\right)^{\mathrm{T}}Z\left(\bar{L}_2^1-\tilde{L}_1^1\right)\end{aligned} \tag{38}$$

where ω, $\bar{\omega}$ and $\bar{\varXi}_1$ are defined in (35c)–(35e). Then, using Schur complement Lemma, the following is yielded:

$$\tilde{\Psi}_1+\tilde{\Psi}_2+\tilde{\Psi}_3 \le \begin{bmatrix} \omega+\bar{\omega} & \left(\tilde{L}_1^1\right)^{\rm T} & \sqrt{d-1}\left(\bar{L}_2^1-\tilde{L}_1^1\right)^{\rm T} & \sqrt{d}\left(\tilde{L}_1^1-L_0^1\right)^{\rm T} & \Xi_1^{\rm T} \\ * & -Q^{-1} & 0 & 0 & 0 \\ * & * & -Z^{-1} & 0 & 0 \\ * & * & * & -Z^{-1} & 0 \\ * & * & * & * & -P^{-1} \end{bmatrix}. \tag{39}$$

Now, substituting $\tilde{A}=A+\delta A\,(k)$ and $\tilde{A}_d=A_d+\delta A_d\,(k)$ into $\tilde{L}_1^1$ leads to $\tilde{L}_1^1=(A+\delta A\,(k))\,e_1+(A_d+\delta A_d\,(k))\,e_2$. By rearranging some terms, $\tilde{L}_1^1$ is rewritten in the form of

$$\tilde{L}_1^1=\ L_1^1+\delta L_1^1 \tag{40}$$

where $L_1^1=Ae_1+A_de_2$ and $\delta L_1^1=\delta A\,(k)\,e_1+\delta A_d\,(k)\,e_2$ denote the certain and uncertain parts, respectively. Thus, using (40), the term $\tilde{\Psi}_1+\tilde{\Psi}_2+\tilde{\Psi}_3$ in (39) can be separated as follows:

$$\tilde{\Psi}_1+\tilde{\Psi}_2+\tilde{\Psi}_3\le\psi+\delta\psi \tag{41}$$

in which:

$$\delta\psi=\begin{bmatrix} 0 & \delta L_1^{1\,\rm T} & -\sqrt{d-1}\delta L_1^{1\,\rm T} & \sqrt{d}\delta L_1^{1\,\rm T} & \begin{bmatrix} 0 & \delta L_1^{1\,\rm T}\end{bmatrix} \\ \delta L_1^1 & 0 & 0 & 0 & \begin{bmatrix} 0 & 0\end{bmatrix} \\ -\sqrt{d-1}\delta L_1^1 & 0 & 0 & 0 & \begin{bmatrix} 0 & 0\end{bmatrix} \\ \sqrt{d}\delta L_1^1 & 0 & 0 & 0 & \begin{bmatrix} 0 & 0\end{bmatrix} \\ \begin{bmatrix} 0 \\ \delta L_1^1\end{bmatrix} & \begin{bmatrix} 0 \\ 0\end{bmatrix} & \begin{bmatrix} 0 \\ 0\end{bmatrix} & \begin{bmatrix} 0 \\ 0\end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0\end{bmatrix} \end{bmatrix} \tag{42a}$$

$$\psi=\begin{bmatrix} \omega+\delta\omega & \left(L_1^1\right)^{\rm T} & \sqrt{d-1}\left(\bar{L}_2^1-L_1^1\right)^{\rm T} & \sqrt{d}\left(L_1^1-L_0^1\right)^{\rm T} & \Xi_1^{\rm T} \\ * & -Q^{-1} & 0 & 0 & 0 \\ * & * & -Z^{-1} & 0 & 0 \\ * & * & * & -Z^{-1} & 0 \\ * & * & * & * & -P^{-1} \end{bmatrix}. \tag{42b}$$

Therefore, (36) holds if for any $\xi_1\in\mathbf{R}^{4n}$ and $\xi_i\in\mathbf{R}^n$, $i=2,\cdots,6$, $\bar{\xi}=\left[\xi_1^{\rm T},\ \xi_2^{\rm T},\xi_3^{\rm T},\xi_4^{\rm T},\xi_5^{\rm T},\xi_6^{\rm T},\right]^{\rm T}$, $\xi_i\neq 0$, we have

$$\bar{\xi}^{\rm T}\left(\psi+\delta\psi\right)\bar{\xi}=\bar{\xi}^{\rm T}\psi\bar{\xi}+\bar{\xi}^{\rm T}\delta\psi\bar{\xi}<0 \tag{43}$$

Then,

$$\bar{\xi}^{\rm T}\psi\bar{\xi}<-\bar{\xi}^{\rm T}\delta\psi\bar{\xi}. \tag{44}$$

Considering $\delta\psi$ from (42a), we have

$$\begin{aligned}\bar{\xi}^{\rm T}\delta\psi\bar{\xi}=&-2({\xi_1}^{\rm T}\delta{L_1^1}^{\rm T}\xi_2-\sqrt{d-1}{\xi_1}^{\rm T}\delta{L_1^1}^{\rm T}\xi_3+\\&\sqrt{d}{\xi_1}^{\rm T}\delta{L_1^1}^{\rm T}\xi_4+{\xi_1}^{\rm T}\delta{L_1^1}^{\rm T}\xi_6).\end{aligned} \tag{45}$$

Replacing δL_1^1 from (40) and using (32), (44) and (45) implies that

$$\begin{aligned}\bar{\xi}^{\rm T}\psi\bar{\xi}<&-2\max\{{\xi_1}^{\rm T}(N_Ae_1+N_de_2)^{\rm T}(F^{\rm T}\,(k)\,M^{\rm T}\xi_2-\\&\sqrt{d-1}F^{\rm T}\,(k)\,M^{\rm T}\xi_3+\sqrt{d}F^{\rm T}\,(k)\,M^{\rm T}\xi_4+\\&F^{T}\,(k)\,M^{\rm T}\xi_6)|F^{\rm T}\,(k)\,F\,(k)\le I\}\le 0\end{aligned} \tag{46}$$

Equivalently,

$$\begin{aligned}(\bar{\xi}^{\rm T}\psi\bar{\xi})^2>&\,4\max\{[{\xi_1}^{\rm T}(N_Ae_1+N_de_2)^{\rm T}(F^{\rm T}(k)M^{\rm T}\xi_2-\\&\sqrt{d-1}F^{\rm T}(k)M^{\rm T}\xi_3+\sqrt{d}F^{\rm T}(k)M^{\rm T}\xi_4+\\&F^{\rm T}(k)M^{\rm T}\xi_6)]^2|\quad F^{\rm T}(k)F(k)\le I\}.\end{aligned} \tag{47}$$

Using Lemma A.2 in [27], the following inequality holds:

$$\begin{aligned}&\left(\bar{\xi}^{\rm T}\psi\bar{\xi}\right)^2>4\left[{\xi_1}^{\rm T}(N_Ae_1+N_de_2)^{\rm T}\,(N_Ae_1+N_de_2)\,\xi_1\right]\times\\&[(F^{\rm T}\,(k)\,M^{\rm T}\xi_2-\sqrt{d-1}F^{\rm T}\,(k)\,M^{\rm T}\xi_3+\sqrt{d}F^{\rm T}\,(k)\,M^{\rm T}\xi_4+\\&F^{\rm T}\,(k)\,M^{\rm T}\xi_6)^{\rm T}(F^{\rm T}\,(k)\,M^{\rm T}\xi_2-\sqrt{d-1}F^{\rm T}\,(k)\,M^{\rm T}\xi_3+\\&\sqrt{d}F^{\rm T}\,(k)\,M^{\rm T}\xi_4+F^{\rm T}\,(k)\,M^{\rm T}\xi_6)|F^{\rm T}\,(k)\,F\,(k)\le I].\end{aligned} \tag{48}$$

In recent inequality, considering $F^{\rm T}(k)F(k)\le I$ yields:

$$\left(\bar{\xi}^{\rm T}\psi\bar{\xi}\right)^2 - 4[\xi_1]^{\rm T}[(N_Ae_1+N_de_2)^{\rm T}(N_Ae_1+N_de_2)]\,[\xi_1]\begin{bmatrix}\xi_2\\ \xi_3\\ \xi_4\\ \xi_6\end{bmatrix}^{\rm T}\begin{bmatrix} MM^{\rm T} & -\sqrt{d-1}MM^{\rm T} & \sqrt{d}MM^{\rm T} & MM^{\rm T}\\ -\sqrt{d-1}MM^{\rm T} & (d-1)\,MM^{\rm T} & -\sqrt{d\,(d-1)}MM^{\rm T} & -\sqrt{d-1}MM^{\rm T}\\ \sqrt{d}MM^{\rm T} & -\sqrt{d\,(d-1)}MM^{\rm T} & dMM^{\rm T} & \sqrt{d}MM^{\rm T}\\ MM^{\rm T} & -\sqrt{d-1}MM^{\rm T} & \sqrt{d}MM^{\rm T} & MM^{\rm T}\end{bmatrix}\begin{bmatrix}\xi_2\\ \xi_3\\ \xi_4\\ \xi_6\end{bmatrix} > 0. \tag{49}$$

By using Lemma A.3 in [27], there exists a constant scalar $\varepsilon_5 > 0$ such that:

$$\psi + \varepsilon_5^{-1}\begin{bmatrix}(N_Ae_1+N_de_2)^{\rm T}(N_Ae_1+N_de_2) & 0_{4n\times 5n}\\ 0_{5n\times 4n} & 0_{5n\times 5n}\end{bmatrix} + \varepsilon_5\begin{bmatrix} 0_{4n\times 4n} & 0_{4n\times n} & 0_{4n\times n} & 0_{4n\times n} & \begin{bmatrix}0_{4n\times n} & 0_{4n\times n}\end{bmatrix}\\ 0_{n\times 4n} & \lambda & -\sqrt{d-1}\lambda & \sqrt{d}\lambda & \begin{bmatrix}0_{n\times n} & \lambda\end{bmatrix}\\ 0_{n\times 4n} & -\sqrt{d-1}\lambda & (d-1)\lambda & -\sqrt{d(d-1)}\lambda & \begin{bmatrix}0_{n\times n} & -\sqrt{d-1}\lambda\end{bmatrix}\\ 0_{n\times 4n} & \sqrt{d}\lambda & -\sqrt{d(d-1)}\lambda & d\lambda & \begin{bmatrix}0_{n\times n} & \sqrt{d}\lambda\end{bmatrix}\\ \begin{bmatrix}0_{n\times 4n}\\ 0_{n\times 4n}\end{bmatrix} & \begin{bmatrix}0_{n\times n}\\ \lambda\end{bmatrix} & \begin{bmatrix}0_{n\times n}\\ -\sqrt{d-1}\lambda\end{bmatrix} & \begin{bmatrix}0_{n\times n}\\ \sqrt{d}\lambda\end{bmatrix} & \begin{bmatrix}0_{n\times n} & 0_{n\times n}\\ 0_{n\times n} & \lambda\end{bmatrix}\end{bmatrix} < 0 \tag{50}$$

where λ is defined in (35h). Then, (34) is obtained by some manipulations, and the proof is completed. □

Remark 4. The robust stability condition (34) contains Q^{-1}, Z^{-1}, P^{-1}, ε_5^{-1} which causes nonlinearity. Thus, the obtained robust stability condition is not LMI. Therefore, the problem is solved by following the Algorithm 2 which converts the non-convex BMI problem into an LMI-based nonlinear minimization problem[28]. Defining $R_1 = P^{-1}$, $R_2 = Q^{-1}$, $R_3 = Z^{-1}$ and $R_4 = \varepsilon_5^{-1}$, Algorithm 2 can be used for the respected BMI problem (34).

Algorithm 2. Solving BMI problem

Minimize : Trace $\{PR_1 + QR_2 + ZR_3\}$ + Trace $\{\varepsilon_5 R_4\}$
subject to (34) and
$P > 0, \quad Q > 0, \quad Z > 0, \quad \varepsilon_i > 0 (i = 1, \cdots, 5)$
$\begin{bmatrix} P & I\\ * & R_1\end{bmatrix} > 0, \begin{bmatrix} Q & I\\ * & R_2\end{bmatrix} > 0, \begin{bmatrix} Z & I\\ * & R_3\end{bmatrix} > 0,$
$\begin{bmatrix} \varepsilon_5 & 1\\ * & R_4\end{bmatrix} > 0.$

3.3 Stabilization based on non-monotonic Lyapunov-Krasovskii method

The NMLK stability condition for state-delay systems given in Section 3.1 can be applied to design a state feedback controller for the input/output delay systems. Assume the following discrete-time input-delay system:

$$\begin{cases} x\,(k+1) = Ax\,(k) + Bu(k-d)\\ y\,(k) = Cx\,(k).\end{cases} \tag{51}$$

The control input with an output feedback controller is assumed as

$$u\,(k) = -Fy\,(k) + r\,(k) \tag{52}$$

where $r\,(k)$ is the input reference. Using (52), the resulting closed-loop system is derived as a state-delay system shown in (53):

$$x\,(k+1) = Ax\,(k) - BFCx\,(k-d) + Br\,(k)\,. \tag{53}$$

The following result will present a less conservative technique to design the controller gain F.

Theorem 4. The closed-loop input delay system (53) is globally asymptotically 2-step non-monotonic stabilizable if there exists a controller gain vector $F \in \mathbf{R}^{n\times 1}$ and positive definite matrices $P \in \mathbf{R}^{2n\times 2n}$, $Q \in \mathbf{R}^{n\times n}$ and $Z \in \mathbf{R}^{n\times n}$ such that the following holds:

$$\begin{bmatrix} \sum_1 & (L_1^1)^{\rm T} & \begin{bmatrix}(L_1^2)^{\rm T} & (L_0^2)^{\rm T}\end{bmatrix} & \sqrt{d-1}(L_1^1 - L_0^1)^{\rm T} & \sqrt{d}(L_2^1 - L_1^1)^{\rm T} & \Xi_1^{\rm T}\\ * & -Q^{-1} & 0 & 0 & 0 & 0\\ * & * & \sum_2 & 0 & 0 & 0\\ * & * & * & -Z^{-1} & 0 & 0\\ * & * & * & * & -Z^{-1} & 0\\ * & * & * & * & * & -P^{-1}\end{bmatrix} < 0 \tag{54}$$

where $\tilde{A} = A$ and $\tilde{A}_d = -BFC$ and

$$\sum_1 = -\Xi_2^{\mathrm{T}}P\Xi_2 - \frac{2}{d}\varphi_2{}^{\mathrm{T}}Z\varphi_2 - \frac{6\left(d-1\right)}{d\left(d+1\right)}\Pi_2^{\mathrm{T}}Z\Pi_2 - \left(L_0^1\right)^{\mathrm{T}}QL_0^1 - \left(L_0^2\right)^{\mathrm{T}}QL_0^2 \quad (55a)$$

$$\sum_2 = \begin{bmatrix} -Q^{-1} & -Q^{-1} \\ -Q^{-1} & Q^{-1}-Z^{-1} \end{bmatrix} \quad (55b)$$

$$\Xi_1 = \begin{bmatrix} L_2^1 \\ L_0^1+L_1^1-L_1^2+e_4 \end{bmatrix}, \Xi_2 = \begin{bmatrix} e_1 \\ \left(L_0^2+e_4\right) \end{bmatrix}$$

$$\varphi_2 = \left(L_0^1-L_0^2\right), \qquad \Pi_2 = L_0^1+L_0^2-\frac{2}{d-1}e_4 \quad (55c)$$

$$\begin{aligned} &L_0^1 = e_1, \;\; L_1^1 = \tilde{A}e_1+\tilde{A}_de_2, \\ &L_2^1 = \tilde{A}^2e_1+\left(\tilde{A}\tilde{A}_d+\tilde{A}_d\tilde{A}\right)e_2+\tilde{A}_d^2e_3 \\ &L_0^2 = e_2, \;\; L_1^2 = \tilde{A}e_2+\tilde{A}_de_3. \end{aligned} \quad (55d)$$

Proof. System (53) is in the form of system (1), having an extra input term. This term has not any effect on the stability. So the proof can be carried out with a similar procedure to the proof of Theorem 2. More specifically, ΔV_1 and ΔV_2 are similar to the ones in the Theorem 2, but ΔV_3 is calculated as follows:

$$\begin{aligned} \Delta_2V_3 = {} & \xi^{\mathrm{T}}(x)((d-1)(L_1^1-L_0^1)^{\mathrm{T}}Z(L_1^1-L_0^1)+ \\ & d(L_2^1-L_1^1)^{\mathrm{T}}Z(L_2^1-L_1^1)+ \\ & (L_1^2-L_0^2)^{\mathrm{T}}Z(L_1^2-L_0^2))\xi(x)-2\sum_{j=k-d}^{k-1}\eta^{\mathrm{T}}(j)Z\eta(j). \end{aligned} \quad (56)$$

Using Lemma 1 for the summation term, we have

$$\begin{aligned} &\Delta_2V_3 < \xi^{\mathrm{T}}(x)((d-1)(L_1^1-L_0^1)^{\mathrm{T}}Z(L_1^1-L_0^1)+ \\ &d(L_2^1-L_1^1)^{\mathrm{T}}Z(L_2^1-L_1^1)+(L_1^2-L_0^2)^{\mathrm{T}}Z(L_1^2-L_0^2)- \\ &\frac{2}{d}\varphi_2{}^{\mathrm{T}}Z\varphi_2-\frac{6(d-1)}{d(d+1)}\Pi_2^{\mathrm{T}}Z\Pi_2)\xi(x) \end{aligned} \quad (57)$$

where $\varphi_2 = \left(L_0^1-L_0^2\right)$ and $\Pi_2 = L_0^1+L_0^2-\frac{2}{d-1}e_4$. Finally, $\Psi_1+\Psi_2+\Psi_3<0$ is attained for the stabilization, as follows:

$$\begin{aligned} &\Psi_1+\Psi_2+\Psi_3 \leq \Xi_1^{\mathrm{T}}P\Xi_1-\Xi_2^{\mathrm{T}}P\Xi_2+\left(L_0^1\right)^{\mathrm{T}}QL_0^1+ \\ &\left(L_1^1\right)^{\mathrm{T}}QL_1^1-\left(L_0^2\right)^{\mathrm{T}}QL_0^2-\left(L_1^2\right)^{\mathrm{T}}QL_1^2+(d- \\ &1)\left(L_1^1-L_0^1\right)^{\mathrm{T}}Z\left(L_1^1-L_0^1\right)+d\left(L_2^1-L_1^1\right)^{\mathrm{T}}Z(L_2^1- \\ &L_1^1)+\left(L_1^2-L_0^2\right)^{\mathrm{T}}Z\left(L_1^2-L_0^2\right)-\frac{2}{d}\varphi_2{}^{\mathrm{T}}Z\varphi_2- \\ &\frac{6\left(d-1\right)}{d\left(d+1\right)}\Pi_2^{\mathrm{T}}Z\Pi_2<0 \end{aligned} \quad (58)$$

where the parameters are defined in (55c) to (55d). By some manipulations, (58) can be written as follows:

$$\begin{aligned} &\Xi_1^{\mathrm{T}}P\Xi_1-\Xi_2^{\mathrm{T}}P\Xi_2+\left(L_0^1\right)^{\mathrm{T}}QL_0^1+\left(L_1^1\right)^{\mathrm{T}}QL_1^1+ \\ &(d-1)\left(L_1^1-L_0^1\right)^{\mathrm{T}}Z\left(L_1^1-L_0^1\right)-\left(L_0^2\right)^{\mathrm{T}}QL_0^2+ \\ &d\left(L_2^1-L_1^1\right)^{\mathrm{T}}Z\left(L_2^1-L_1^1\right)+ \\ &\begin{bmatrix} L_1^2 \\ L_0^2 \end{bmatrix}^{\mathrm{T}}\begin{bmatrix} Z-Q & -Z \\ -Z & Z \end{bmatrix}\begin{bmatrix} L_1^2 \\ L_0^2 \end{bmatrix}-\frac{2}{d}\varphi_2{}^{\mathrm{T}}Z\varphi_2- \\ &\frac{6\left(d-1\right)}{d\left(d+1\right)}\Pi_2^{\mathrm{T}}Z\Pi_2<0. \end{aligned} \quad (59)$$

Finally, using the well-known Schur complement Lemma, condition (54) is attained. □

Remark 5. Converting the term $\left(L_1^2-L_0^2\right)^{\mathrm{T}}Z\left(L_1^2-L_0^2\right)-\left(L_0^2\right)^{\mathrm{T}}QL_0^2$ in (58) to $\begin{bmatrix} L_1^2 \\ L_0^2 \end{bmatrix}^{\mathrm{T}}\begin{bmatrix} Z-Q & -Z \\ -Z & Z \end{bmatrix}\begin{bmatrix} L_1^2 \\ L_0^2 \end{bmatrix}$ in (59) is to reduce the conservatism. The term $-\left(L_1^2\right)^{\mathrm{T}}QL_1^2$ is negative definite and cannot be used in Schur complement. Also omiting this term can increase the conservatism. On the other hand, $\begin{bmatrix} Z-Q & -Z \\ -Z & Z \end{bmatrix}$ is the only form that its obtained inverse in using Schur complement lemma (the parameter $\sum_2$ in (55b)) is linear and does not convert the condition to BMI.

Remark 6. The obtained stabilization conditions in Theorem 4 are BMI due to the term L_2^1, and definitions of $\tilde{A}$ and $\tilde{A}_d$. Therefore, by defining the new variables $\tilde{F}=BFC$, $U_1=\tilde{F}^2$, $\tilde{H}=BH$, $U_2=\tilde{F}\tilde{H}$, the matrix L_2^1 can be written as (60), which removes the second order terms.

$$\begin{aligned} L_2^1 = {} & \begin{bmatrix} A^2-A\tilde{F}-\tilde{F}A-U_1-\tilde{H}C & A\tilde{H}-U_2+\tilde{H} \\ -CA+C\tilde{F}-C & -C\tilde{H}+I_{r\times r} \end{bmatrix}e_1+ \\ & \begin{bmatrix} -A\tilde{F}-\tilde{F}A+2U_1-\tilde{H}C & \tilde{F}\tilde{H} \\ 2C\tilde{F}-C+CA & -C\tilde{H} \end{bmatrix}e_2+ \\ & \begin{bmatrix} U_1 & 0 \\ C\tilde{F} & 0 \end{bmatrix}e_3. \end{aligned} \quad (60)$$

Then, F and H can be easily obtained using calculated $\tilde{F}=BFC$ and $\tilde{H}=BH$. Also, due to the presence of inverse parameters (Q^{-1},Z^{-1} and P^{-1}) in (55b) and (54), the derived conditions in Theorem 4 are not LMI. Thus, Algorithm 2 in Remark 4 can be used.

Remark 7. Having different connected process units causes various streams of materials in some large-scale industrial plants. Model (61) represents such processes:

$$\begin{cases} x\left(k+1\right)=Ax\left(k\right)+Bu\left(k\right) \\ y\left(k\right)=Cx\left(k\right)+Cx\left(k-d\right). \end{cases} \quad (61)$$

Applying an output feedback controller to this process ends to a state-delay closed loop system. Moreover, using an integrator can eliminate the steady state error.

If we show the output error with $e(k)$, then the integrator dynamic model of the controller is

$$e(k+1) = e(k) + (r(k) - y(k)). \qquad (62a)$$

Using the state space representation (61) in (62a), we have $e(k+1) = e(k) + (r(k) - Cx(k) - Cx(k-d))$, then the feedback control law is assumed as

$$u(k) = He(k) - Fy(k) + r(k) \qquad (62b)$$

in which F and H are output feedback controller and integrator gains, respectively. $r(k)$ is the reference input. Therefore, the closed loop augmented system is derived as (62c):

$$\begin{bmatrix} x(k+1) \\ e(k+1) \end{bmatrix} = \begin{bmatrix} A - BFC & BH \\ -C & I_{r\times r} \end{bmatrix} \begin{bmatrix} x(k) \\ e(k) \end{bmatrix} + \begin{bmatrix} -BFC & 0 \\ -C & 0 \end{bmatrix} \begin{bmatrix} x(k-d) \\ e(k-d) \end{bmatrix} + \begin{bmatrix} B \\ I_{r\times r} \end{bmatrix} r(k). \qquad (62c)$$

The controller gain F can be designed as stated in Corollary 2.

Corollary 2. The closed-loop time-delay system (62c) is globally asymptotically 2-step non-monotonic stabilizable if there exist a controller gain $F \in \mathcal{R}^{n\times 1}$, an integrator gain $H \in \mathbf{R}^{1\times 1}$ and positive definite matrices $P \in \mathbf{R}^{2n\times 2n}$, $Q \in \mathbf{R}^{n\times n}$ and $Z \in \mathbf{R}^{n\times n}$ such that (54) holds, where $\tilde{A} = \begin{bmatrix} A - BFC & BH \\ -C & I_{r\times r} \end{bmatrix}$ and $\tilde{A}_d = \begin{bmatrix} -BFC & 0 \\ -C & 0 \end{bmatrix}$.

Proof. Considering the system coefficient matrices in (62c), the stabilization conditions are obtained by applying Theorem 4 on (62c). □

4 Illustrative examples

4.1 Numerical examples

This section provides numerical examples evaluating Theorems 2 and 3. In this regard, Examples 1 and 2 investigate the stability analysis of two DTSD systems by Theorem 2. Moreover, using Theorem 3, the robust stability of an uncertain DTSD system is investigated in Example 3. Also, the proposed NMLK based stabilization algorithm in Theorem 4 is evaluated in Example 4.

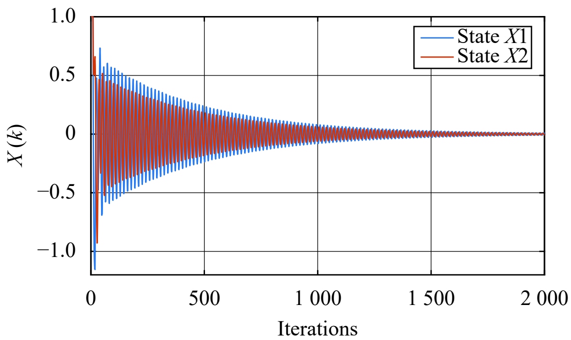

Fig. 1 States of the system (a=0.65)

Example 1. Consider the following DTSD system:

$$X(k+1) = \begin{bmatrix} a & 0.3 \\ -0.1 & 0.7 \end{bmatrix} X(k) + \begin{bmatrix} -0.4 & -0.2 \\ 0.2 & -0.1 \end{bmatrix} X(k-d) \qquad (63)$$

where $a \in \mathbf{R}$. This example investigates calculation of maximum admissible upper bound of delay d using Theorem 2. In this plant, as a increases, maximum admissible d decreases to retain stability. Stability conditions in Theorem 2 are LMIs, which can be solved using YALMIP toolbox in MATLAB software [29]. Table 1 shows the efficacy of Theorem 2 in the stability analysis compared with references [9], [30], and [31].

Table 1 Maximum admissible d

	$a = 0.65$	$a = 1.12$
[30]	5	4
[31]	5	4
[9]	7	4
Theorem 2	8	5

The states of the system are plotted in Fig. 1 with the initial condition $X(-8) = X(-7) = \cdots = X(0) = [1 \ \ -1]^{\mathrm{T}}$, where $a = 0.65$. Fig. 2 shows the NMLK functional, in which non-monotonically decreasing is depicted in the magnified part.

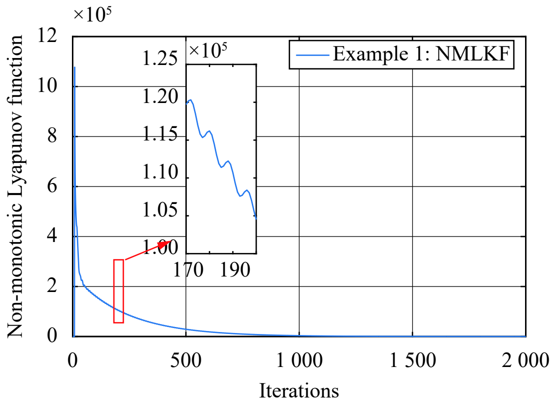

Fig. 2 Non-monotonic Lyapunov functional (a=0.65)

Fig. 3 demonstrates the number of consecutive steps which the NMLK functional locally increases. Since the non-monotonicity step is $m = 8$, increasing steps are less than 8. Most of the increasing steps in this system are 2 and 3.

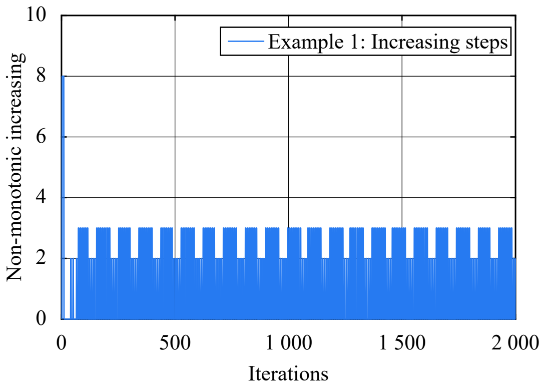

Fig. 3 The increasing steps in NMLKF for (a=0.65)

Example 2. Consider the following state-delay system:

$$X(k+1) = \begin{bmatrix} 0.8 & 0 \\ 0.05 & 0.9 \end{bmatrix} X(k) + \begin{bmatrix} -0.1 & \rho \\ -0.2 & -0.1 \end{bmatrix} X(k-d). \quad (64)$$

Table 2 compares the maximum upper bound of delay d obtained by Theorem 2 in this article, stabilization criteria derived using Lyapunov functional in [5], a finite sum inequality based approach in [32], utilization of zero equalities in [33] and Abel lemma based method in [9], for two values of ρ. The first 3 rows show that methods presented in [5], [32] and [33] cannot prove the stability for $\rho = 0.056$, while [9] provides a reasonably good stability margin. To apply Theorem 2, m is selected to be 57. For both values of ρ, Theorem 2 provides less conservative margin compared to all others.

Table 2 Maximum upper bound of d

	$\rho = 0$	$\rho = 0.056$
[5]	17	-
[32]	18	-
[33]	19	-
[9]	4430	57
Theorem 2	∞	58

Example 3. Consider the following uncertain DTSD system:

$$X(k+1) = (A + \delta A) X(k) + (A_d + \delta A_d) X(k-7) \quad (65)$$

in which considering the uncertainty matrices (32):

$$A = \begin{bmatrix} -0.8 & -0.5 \\ 0.4 & -0.8 \end{bmatrix}, \ A_d = \begin{bmatrix} 0.3 & 0.1 \\ -0.1 & -0.1 \end{bmatrix},$$

$$M = \begin{bmatrix} 0.3 \\ 0.1 \end{bmatrix}$$

$$N_A = \begin{bmatrix} 0.15 & 0.1 \end{bmatrix}, \ N_d = \begin{bmatrix} 0.2 & 0.1 \end{bmatrix}.$$

The robust stability of system (65) is investigated using Theorem 3 and Remark 4. The trend of the robust NMLK functional is plotted in Fig. 4 with the initial condition $X(-7) = X(-6) = \cdots = X(0) = [10 \quad 10]^{\rm T}$. As it is demonstrated in Fig. 4, sometimes the functional increases for one step, but the trend of that is decreasing.

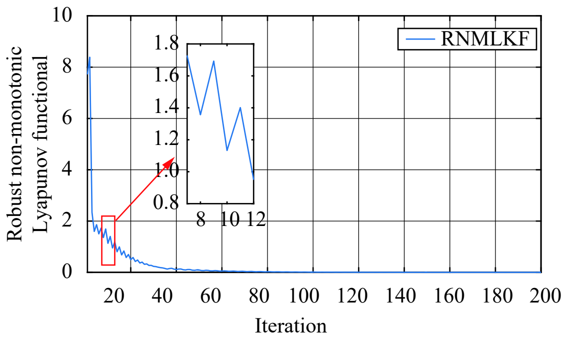

Fig. 4 Robust non-monotonic Lyapunov functional of Example 3

Example 4. A first order plus time delay (FOPTD) model of a coupled tank apparatus is obtained in [34] with a transfer function of $G(s) = \dfrac{5.15}{109s+1} {\rm e}^{-4.5s}$. The discrete state space representation of $G(s)$ with sample time $T = 0.1$s is as follows:

$$\begin{cases} X(k+1) = {\rm A}X(k) + {\rm B}u(k - {\rm d}) \\ Y(k) = CX(k) \end{cases} \quad (66)$$

with $A = 0.999\,1$ $B = 0.099\,9$, $C = 0.047\,2$. Considering $e(k)$ and $u(k)$ as (62a) and (62b), respectively, the state space representation of the closed-loop augmented controller system is achieved as follows:

$$\begin{bmatrix} X(k+1) \\ e(k+1) \end{bmatrix} = \tilde{A} \begin{bmatrix} X(k) \\ e(k) \end{bmatrix} + \tilde{A}_d \begin{bmatrix} X(k-d) \\ e(k-d) \end{bmatrix} \quad (67)$$

where $\tilde{A} = \begin{bmatrix} A & 0 \\ -C & I_{r\times r} \end{bmatrix}$ and $\tilde{A}_d = \begin{bmatrix} -BFC & BH \\ 0 & 0 \end{bmatrix}$.

Therefore, using Remark 6 and applying Theorem 4 with the non-monotonicity step $m = 2$, the controller gains are obtained as $F = 9.8435$ and $H = 0.0301$. Fig. 5 compares the step response of NMLK based controller and simulation results of first order PI (FOPI) controller designed in [34]. Two disturbances of 1.5 cm and –1.5 cm are applied at t = 600 s and t = 1 300 s, respectively. Fig. 5 illustrates that NMLK based controller performs better in rise time, overshoot and disturbance rejections.

5 Implementation on pH neutralization process

The pH neutralizing pilot plant is a highly non-linear process shown in Fig. 6. In this section, a controller is designed based on Corollary 2 for a pH neutralizing pilot process. Then, simulation and experimental implementa-

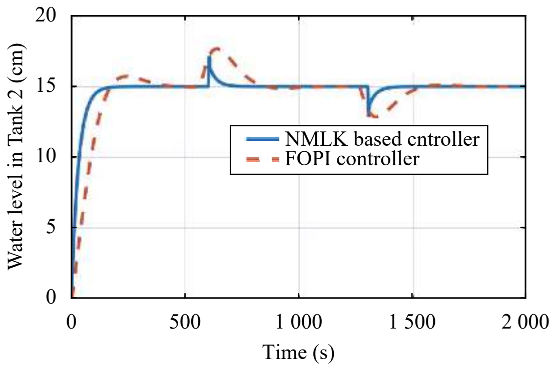

Fig. 5 Step response of the coupled tank with ±10% disturbance

Fig. 6 Neutralization pH process plant

tion are presented.

5.1 pH neutralization process

The pH neutralizing process is a nonlinear plant with a highly challenging control problem. Figs. 6 and 7 show the picture and flow diagram of the understudy pH neutralization pilot plant. It consists of three inlet stream of tap water, acetic acid (CH3COOH as acid), sodium hydroxide (base) materials, a continuous stirred tank reactor (CSTR), three dosing pump, an outlet pump, two outlet pipes for the effluent streams and a pH sensor. Acid, base, and water are injected into the CSTR by corresponding dosing pumps[35, 36], where there is a motorized mixer in order to have a well-mixed liquid. The CSTR content is pumped out of the CSTR using an outlet pump. The output of the pump can be transport through two outlet pipes with different delay times: a direct path and a delayed path. The delayed path imposes transportation delay to the process. These two streams are combined again, where the pH sensor is located. The proportion of the flow through each of the paths can be adjusted using a three-way valve.

The block diagram of the control system is illustrated in Fig. 8. The pH of the combined output stream is controlled by the flow rate of the base inlet, while the flow rate of the acetic acid is constant. Tap water is added to the process to control the level of the reactor. The

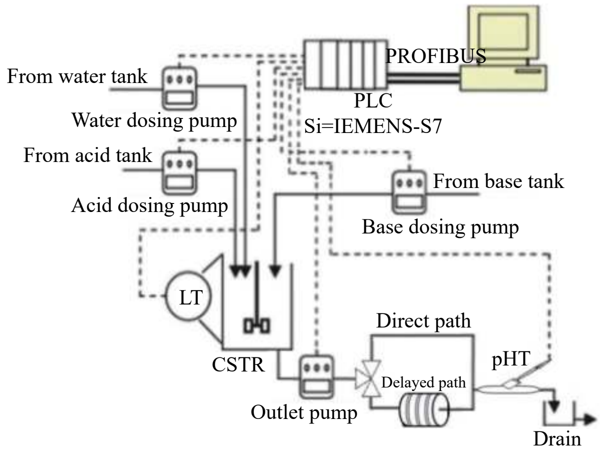

Fig. 7 Schematic diagram of the pH neutralization process

effluent stream flow rate is kept constant and equally divided between the two outlet pipes. The pH of the effluent stream is controlled using output feedback with an integral controller. The dynamic model of the augmented closed loop control system is as explained in Remark 7.

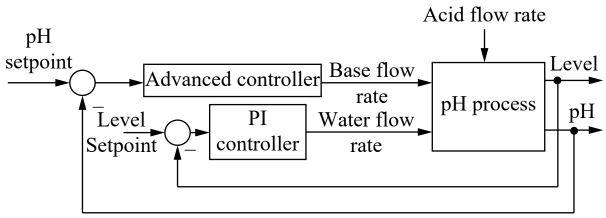

Fig. 8 Control loops in pH neutralization process

In our experiment, acid and output stream flow rates have been set to 0.45 mlit/s and 1.45 mlit/s, respectively. The concentration of the acid solution has been considered 1.5 mlit/lit which is led to the acid pH of 3. Moreover, the base concentration is 1.0 gr/lit that is led to a base solution with pH=12. The sampling time is 5 s.

In this article, the focus is on designing a controller in order to control the pH value. Therefore, a PI controller with a transfer function $C_l(s) = 0.2 + \frac{20}{s}$ is used as the level controller to maintain the CSTR level in h=11 cm[36].

5.2 Simulation results

In this section, the mentioned pH neutralization process is controlled using the proposed stabilization theorem. A first-order dynamic model is constructed for the plant by applying sequential steps to the open loop system[37]. As was mentioned in the previous sub-section, the effluent stream is the combination of the two outlet path. Thus, the effect of the state appears on the output with two different delays:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) + Cx(k-5) \end{cases} \tag{68}$$

with $A = 0.97$ $B = 0.02$, $C = 1$. The state space representation of the closed-loop controlled system is achieved as (62c). Applying Corollary 2 with the non-monotonicity step $m = 2$, the controller gains are obtained as $F = 14.3438$ and $H = 0.531$ in 11 iterations, as represented in Fig. 9.

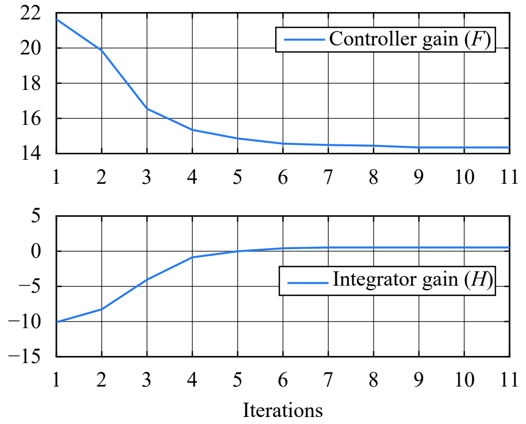

Fig. 9 The convergence procedure of the controller gain (F) and integrator gain (H)

Moreover, the stabilizing matrices P, Q and Z are as following:

$$P = \begin{bmatrix} 0.0074 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0074 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0074 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0074 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0.0090 & 0.0008 \\ 0.0008 & 0.0038 \end{bmatrix}, \quad Z = \begin{bmatrix} 0.004 & 0.0 \\ 0.0 & 0.004 \end{bmatrix}. \qquad (69)$$

In the first step, the closed-loop system is simulated by the above designed controller. The closed-loop response and the control signal (base feed rate) are shown in Figs. 10 and 11, in which a zero-mean Gaussian white noise with variance 0.01 is added to the output of the process.

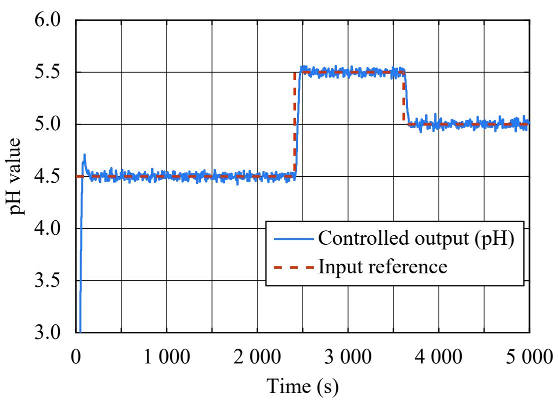

Fig. 10 Reference tracking response

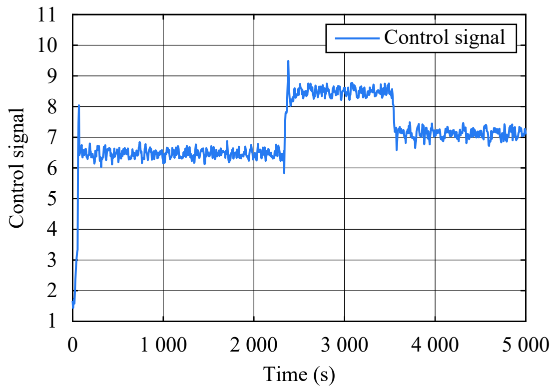

Fig. 11 Control signal (Base feed rate)

Fig. 10 illustrates the reference tracking of the control system. The reference signal consists of 3 different steps around pH=4.5 to 5.5. The output of the system, which is the pH of the CSTR, follows the reference precisely. The controlled output tracks the reference fast enough with vary small overshoots. Also there are some reasonable overshoots in control signal.

The trend of the Lyapunov-Krasovskii functional candidate is depicted in Fig. 12. Non-monotonic decreasing is obvious in this plot, especially in the early time steps. However, as expected, the incremental steps are less than non-monotonicity step $m = 2$.

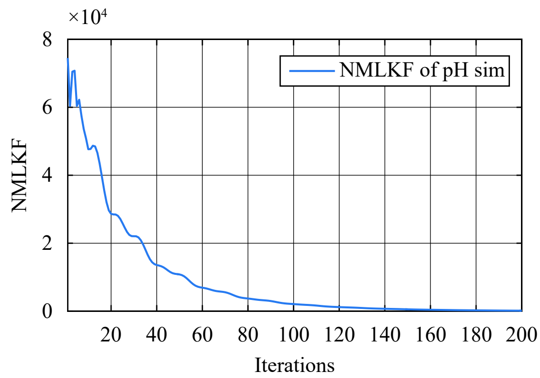

Fig. 12 Non-Monotonic Lyapunov-Krasovskii functional

5.3 Experimental results

An experimental test was conducted on the mentioned pH neutralizing process. There are many challenging issues in this plant that are fascinating for the researchers in the process control systems such as 1) being extremely nonlinear, 2) containing a wide source of disturbances and noise, 3) the pH of the water is considered to be 7 in the models, but it measured to be 7.8 during the experiment, 4) uncertainty in the concentration of the materials inadvertently every time that they are manually prepared, 5) slowly acid neutralization during the time, 6) changes in the temperature of the room, 7) measurement noise in the pH sensor. These practical facts in a pH neutralization process make this apparatus one of the most challenging problems in the process control study. So, we try to reduce the effect of some of the above problems in our experiment. To keep the process maintained in a less nonlinear operating area, the reference is in the pH value of $[4.5, 5.5]$[36]. We try to maintain the duration of the experiment short enough to prevent any needs to extra material preparation. The pH neutralization process is controlled using the proposed NMLK based stabilization method. The designed output feedback controller in Section 5.2 is applied to this plant. The combination of direct and delayed paths constructs the effluent stream in the CSTR, which the pH value is measured. The reference following is depicted in Fig. 13.

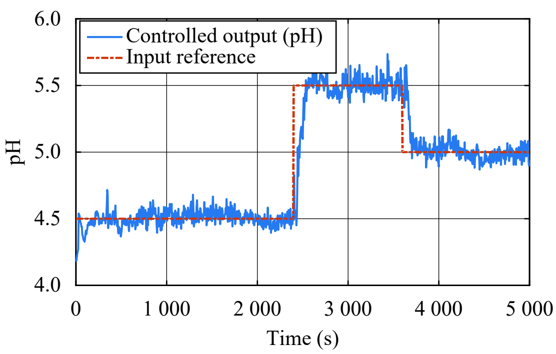

Fig. 13 Reference tracking response in the experimental study of pH neutralization

Although there is the effect of noise and disturbances in the output response, the pH of the CSTR tracks the reference with a practically overdamped response. Fig. 14 illustrates the control signal, which is the base feed rate. The dynamic of the pH neutralization process is highly dependent on the volume of the liquid in the CSTR. The level of the liquid in the CSTR is depicted in Fig. 15. It is seen that the level is kept reasonably constant.

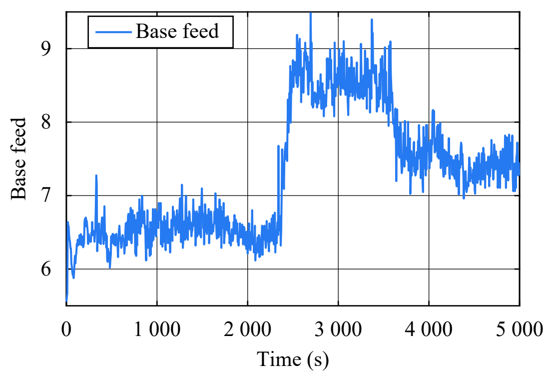

Fig. 14 Control signal (Base feed rate) in the experimental study of pH neutralization

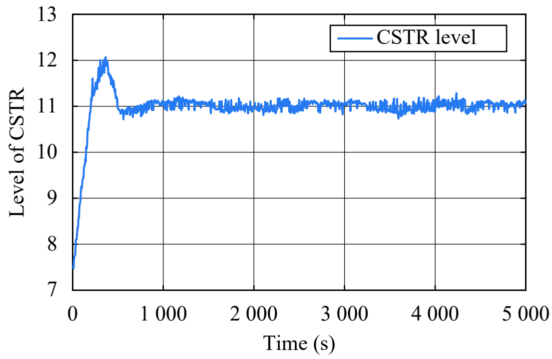

Fig. 15 Level of CSTR in pH process plant

6 Discussions

NMLK based theorems in this paper were proposed for stability, robust stability and stabilization of state-delay systems. Mathematical manipulations in proofs are not routine and need some careful considerations. Some key issues in the extraction of the proposed stability and stabilization LMIs are mentioned in the following:

In the proposed NMLK stability approach, there were many difficulties in calculating ΔV_2 and ΔV_3. It needed many manipulations to reach an appropriate form. Furthermore, the term Γ in (23) was too important to deal with. It is a negative definite term and it could be easily omitted. But, removing this part could increase the conservatism. A remedy is to find an upper bound for it, which is a challenging step. The upper bound must have two characteristic. It must have as little conservatism as possible. It must be in an appropriate form in order to factorize $\xi^{\mathrm{T}}(x)$. Also, the stabilization conditions were carried out similar to the proof of Theorem 2 for the closed loop time-delay system described by (53). Finally, $\Psi_1 + \Psi_2 + \Psi_3 < 0$ is attained for stabilization. This stabilization condition is not an LMI because of the terms $\tilde{A}$ and $\tilde{A}_d$ which contain the controller gain F multiplied by the unknown matrices P, Q or Z. It is expected that using Schur complement will lead to LMI. But this is possible only if a carefully designed form of $\xi^{\mathrm{T}}(x)$ is selected. Therefore, the selection of $\xi^{\mathrm{T}}(x)$ should be such that simultaneously have suitable form for factorization in $\Delta_m V_3$ calculations and finding the upper bound of summation term of Γ in (23). Selecting $\xi^{\mathrm{T}}(x)$ as stated in (5a) leads to a stabilization conditions which reduces the conservativeness by replacing the negative definite terms by their upper bound.

1) Some unique manipulations are necessary in the proof of stabilization Theorem 4, e.g., calculating $\sum_2$ in (59) which is explained in Remark 5.

2) Algorithm 1 in Remark 2 is a novel point of view in calculating past states. Using this algorithm simplifies the LMI conditions programing. In robust stability procedure,

the technique of separating the certain and uncertain parts and novel lemmas were provided.

3) Although many researches address state-delay systems stabilization, there are not many state-delay systems in nature. However, input delay systems as modeled in (51), are appeared frequently in the industries. The application of Theorem 4 on designing state-feedback controller for these systems, opens a new horizon of application of the NMLK based stabilization in industry.

4) The numerical examples emphasis that the NMLK based theorems are less conservative compared with similar approaches. In addition to numerical simulation examples, experimental implementation on a pH neutralization process is presented for the evaluation of the proposed theorems. This shows that the proposed stabilization theorem is completely applicable and can be used experimentally.

7 Conclusions

The novel non-monotonic Lyapunov-Krasovskii (NMLK) theorem was proposed for stability analysis of DTSD systems. In this theorem, the strictly decreasing trend is not necessary for the Lyapunov-Krasovskii functional. Instead, it can increase in a few steps, while the overall trend is decreasing. In this regard, stability and stabilization theorems were presented based on NLMLF. In the proposed NMLK based theorems, a non-monotonicity step is defined, which is the upper bound for admissible incremental steps in NMLK functional. Increasing the non-monotonicity step can directly lessen conservatism at the cost of increasing computations. Furthermore, a non-monotonic robust stability theorem was derived for uncertain time-delay systems. Numerical examples showed that conservatism is reduced compared with some other available methods, and the feasible space is enlarged. Stabilization theorem was used to design a stable output feedback controller for a pH neutralizing process. Experimental results illustrate the efficacy of the applied controller.

Appendix A

In this appendix the upper bounds of $\tilde{L}_2^1$ and $\tilde{L}_1^2$ are calculated in Section *Aa*. The upper of $\tilde{\Xi}_1^{\rm T} P \tilde{\Xi}_1$ is calculated in Section *Ab* and the upper bound of $\delta\omega$ is calculated in Section *Ac*.

Aa. Calculating upper bounds *of* $\tilde{L}_2^1$ and $\tilde{L}_1^2$

The parameter $\tilde{L}_2^1$ can be written in the following form:

$$\tilde{L}_2^1 = \tilde{L}_{21}^1 e_1 + \tilde{L}_{22}^1 e_2 + \tilde{L}_{23}^1 e_3 = (A+\delta A)^2 e_1 + \\ ((A+\delta A)(A_d+\delta A_d) + (A_d+\delta A_d) \\ (A+\delta A)) e_2 + (A_d+\delta A_d)^2 e_3. \tag{A1}$$

Now, Lemma 2 is introduced.

Lemma 2. Considering the matrix $W \in \mathbf{R}^{n\times n}$, the following inequality holds for the spectral norm:

$$W \le ||W|| I_n. \tag{A2}$$

Proof. If x is the eigenvector corresponding to the maximum eigenvalue $\lambda_{\max}(W)$, then $Wx = \lambda_{\max} x$. This yields $|\lambda_{\max}|\, ||x|| \le ||W|| ||x||$, so:

$$|\lambda_{\max}| \le ||W||. \tag{A3}$$

On the other hand, by considering $\mho \in \mathbf{R}^n - \{0\}$ and the properties of eigenvalues, we have $\mho^{\rm T} W \mho \le \lambda_{\max} \mho^{\rm T} \mho$. Therefore, $\mho^{\rm T}(W - \lambda_{\max} I_n)\mho \le 0$. This inequality implies that

$$W \le |\lambda_{\max}|\, I_n. \tag{A4}$$

Therefore, using (A3) and (A4), (A2) is obtained. □

Then, using Lemma 2, we have

$$\tilde{L}_{21}^1 = (A+\delta A)^2 \le ||(A+\delta A)^2|| I_n \le \\ \left(||A||^2 + 2||A|| ||\delta A|| + ||\delta A||^2\right) I_n \tag{A5}$$

$$\tilde{L}_{22}^1 = (A+\delta A)(A_d+\delta A_d) + (A_d+\delta A_d)(A+\delta A) \le \\ 2||(A+\delta A)(A_d+\delta A_d)|| I_n \le \\ 2\left(||AA_d|| + ||A\delta A_d|| + ||\delta A A_d|| + ||\delta A \delta A_d||\right) I_n \tag{A6}$$

$$\tilde{L}_{23}^1 = (A_d+\delta A_d)^2 \le ||(A_d+\delta A_d)^2|| I_n \le \\ \left(||A_d||^2 + 2||A|| ||\delta A_d|| + ||\delta A_d||^2\right) I_n. \tag{A7}$$

Considering (32), it is obvious that $|\delta A||^2 = ||MF(k)N_A||^2$. Then using the properties of spectral norm,

$$||\delta A||^2 = ||MF(k)N_A||^2 \le ||M||^2 ||F(k)||^2 ||N_A||^2. \tag{A8}$$

Using the inequality (33) and considering $||F(k)|| = \sqrt{\lambda_{\max}(F^{\rm T}(k)F(k))}$, we conclude that $||F(k)|| < 1$. So, the following inequality holds:

$$||\delta A||^2 \le ||M||^2 ||N_A||^2. \tag{A9}$$

Similarly, it is simple to show that

$$||\delta A_d|| \le ||M|| ||N_d|| \tag{A10}$$

$$||\delta A_d||^2 \le ||M||^2 ||N_d||^2. \tag{A11}$$

As a result, (A5)–(A7) can be written as

$$\tilde{L}_{21}^1 \le \left(||A||^2 + 2||A|| ||M|| ||N_A|| + ||M||^2 ||N_A||^2\right) I_n \tag{A12}$$

$$\tilde{L}_{23}^1 \leq \left(||A_d||^2 + 2||A_d||||M||||N_d|| + ||M||^2||N_d||^2\right) I_n \quad \text{(A13)}$$

$$\tilde{L}_{22}^1 \leq 2\left(||AA_d|| + ||A\delta A_d|| + ||\delta AA_d|| + ||\delta A\delta A_d||\right) I_n \leq 2(||AA_d|| + ||A||||M||||N_d|| + ||M||||N_A||||A_d||+ ||M||||N_A||||M||||N_d||)I_n. \quad \text{(A14)}$$

Finally, using the calculated upper bounds of $\tilde{L}_{21}^1$, $\tilde{L}_{22}^1$ and $\tilde{L}_{23}^1$, the upper bound of $\tilde{L}_2^1$ is obtained as follows:

$$\tilde{L}_2^1 \leq \bar{L}_2^1 = \left(||A||^2 + 2||A||||M||||N_A|| + ||M||^2||N_A||^2\right) e_1 + 2(||AA_d|| + ||A||||M||||N_d|| + ||M||||N_A||||A_d||+ ||M||||N_A||||M||||N_d||)e_2 + \left(||A_d||^2 + 2||A_d||||M||||N_d|| + ||M||^2||N_d||^2\right) e_3. \quad \text{(A15)}$$

Similarly, $\bar{L}_1^2$, the upper bound of $\tilde{L}_1^2$, can be obtained as follows:

$$\bar{L}_1^2 = (||A|| + ||A||||M||||N_d|| + ||M||||N_A||||A_d||+ ||M||||N_A||||M||||N_d||)e_2 + \left(||A_d||^2 + 2||A_d||||M||||N_d|| + ||M||^2||N_d||^2\right) e_3. \quad \text{(A16)}$$

Ab. Calculating the upper bound of $\tilde{\Xi}_1^{\mathrm{T}} P \tilde{\Xi}_1$

$\tilde{\Xi}_1$ is a $2n \times 4n$ matrix. So, we have the equality $\tilde{\Xi}_1^{\mathrm{T}} P \tilde{\Xi}_1 = \tilde{\tilde{\Xi}}_1^{\mathrm{T}} \bar{P} \tilde{\tilde{\Xi}}_1$, in which $\tilde{\tilde{\Xi}}_1$ and $\bar{P}$ are as follows:

$$\tilde{\tilde{\Xi}}_1 = \begin{bmatrix} \tilde{\Xi}_1 \\ 0_{2n\times 4n} \end{bmatrix}, \bar{P} = \begin{bmatrix} P & 0_{2n\times 2n} \\ 0_{2n\times 2n} & 0_{2n\times 2n} \end{bmatrix}. \quad \text{(A17)}$$

Then, considering calculated $\bar{L}_2^1$ and $\bar{L}_1^2$ in (A15) and (A16):

$$\tilde{\tilde{\Xi}}_1 = \begin{bmatrix} \tilde{L}_2^1 & 0 \\ L_0^1 + \tilde{L}_1^1 - \tilde{L}_1^2 + e_4 & 0 \end{bmatrix} \leq \begin{bmatrix} \bar{L}_2^1 & 0 \\ L_0^1 + \tilde{L}_1^1 + \bar{L}_1^2 + e_4 & 0 \end{bmatrix} = \begin{bmatrix} \bar{L}_2^1 & 0 \\ L_0^1 + L_1^1 + \bar{L}_1^2 + e_4 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \delta L_1^1 \end{bmatrix}. \quad \text{(A18)}$$

Then, we have

$$\tilde{\Xi}_1^{\mathrm{T}} P \tilde{\Xi}_1 = \tilde{\tilde{\Xi}}_1^{T} \bar{P} \tilde{\tilde{\Xi}}_1 \leq \left(\begin{bmatrix} \bar{L}_2^1 & 0 \\ L_0^1 + \tilde{L}_1^1 + \bar{L}_1^2 + e_4 & 0 \end{bmatrix}\right)^{\mathrm{T}} \times \bar{P}\left(\begin{bmatrix} \bar{L}_2^1 & 0 \\ L_0^1 + \tilde{L}_1^1 + \bar{L}_1^2 + e_4 & 0 \end{bmatrix}\right) = \left(\begin{bmatrix} \bar{L}_2^1 \\ L_0^1 + L_1^1 + \bar{L}_1^2 + e_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \delta L_1^1 \end{bmatrix}\right)^{\mathrm{T}} \times P\left(\begin{bmatrix} \bar{L}_2^1 \\ L_0^1 + L_1^1 + \bar{L}_1^2 + e_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \delta L_1^1 \end{bmatrix}\right). \quad \text{(A19)}$$

Finally, the upper bound of $\tilde{\Xi}_1^{\mathrm{T}} P \tilde{\Xi}_1$ is derived as follows:

$$\tilde{\Xi}_1^{\mathrm{T}} P \tilde{\Xi}_1 \leq \left(\bar{\Xi}_1 + \delta\Xi_1\right)^{\mathrm{T}} P \left(\bar{\Xi}_1 + \delta\Xi_1\right) \quad \text{(A20)}$$

where $\bar{\Xi}_1 = \begin{bmatrix} \bar{L}_2^1 \\ L_0^1 + L_1^1 + \bar{L}_1^2 + e_4 \end{bmatrix}$ and $\delta\Xi_1 = \begin{bmatrix} 0 \\ \delta L_1^1 \end{bmatrix}$.

Ac. Calculating the upper bound of $\delta\omega$

$\delta\omega$ can be written as follows:

$$\delta\omega = \delta\omega_1 + \delta\omega_2 + \delta\omega_3 + \delta\omega_4 \quad \text{(A21)}$$

in which $\delta\omega_1 = -\dfrac{2}{d-1}\tilde{\varphi}_2^{\mathrm{T}} Z \tilde{\varphi}_2$, $\delta\omega_2 = -\dfrac{6(d-2)}{d(d-1)} \times \tilde{\Pi}_2^{\mathrm{T}} Z \tilde{\Pi}_2$, $\delta\omega_3 = -\left(\tilde{L}_1^2\right)^{\mathrm{T}} Q \tilde{L}_1^2$, and $\delta\omega_4 = -\left(\tilde{L}_1^2 - L_0^2\right)^{\mathrm{T}} \times Z\left(\tilde{L}_1^2 - L_0^2\right)$.

Therefore, it is easily seen that:

$$\delta\omega_1 = -\frac{2}{d-1}\left(L_0^1 - \tilde{L}_1^2\right)^{\mathrm{T}} Z \left(L_0^1 - \tilde{L}_1^2\right). \quad \text{(A22)}$$

Considering $L_0^1 = e_1$ and $\tilde{L}_1^2 = \tilde{A}e_2 + \tilde{A}_d e_3$, we have

$$\delta\omega_1 = -\frac{2}{d-1}(Ae_2 + A_d e_3 - e_1 + MF(k)J_1)^{\mathrm{T}} Z \times (Ae_2 + A_d e_3 - e_1 + MF(k)J_1) \quad \text{(A23)}$$

which $J_1 = N_A e_2 + N_d e_3$. Therefore, by some manipulations:

$$\delta\omega_1 = -\frac{2}{d-1}\Big\{(\varphi_2)^{\mathrm{T}} Z\varphi_2 + (\varphi_2)^{\mathrm{T}} ZMF(k)J_1 + J_1^{\mathrm{T}} F^{\mathrm{T}}(k) M^{\mathrm{T}} Z\varphi_2 + J_1^{\mathrm{T}} F^{\mathrm{T}}(k) M^{\mathrm{T}} ZMF(k)J_1\Big\} \quad \text{(A24)}$$

In this regard, Lemma 3 is introduced to simplify the recent equation.

Lemma 3. Let M, A, F and N be matrices of appropriate dimension in which $F^{\mathrm{T}}F < I_n$. Let Z be a positive definite symmetric matrix satisfying $(M^{\mathrm{T}}ZM - \varepsilon I_n) > 0$ in which $\varepsilon > 0$. Then, the following inequality holds:

$$-A^{\mathrm{T}}ZMFN - N^{\mathrm{T}}F^{\mathrm{T}}M^{\mathrm{T}}ZA - N^{\mathrm{T}}F^{\mathrm{T}}M^{\mathrm{T}}ZMFN \leq A^{\mathrm{T}}ZM\left(M^{\mathrm{T}}ZM - \varepsilon I_n\right)^{-1}M^{\mathrm{T}}ZA - \varepsilon N^{\mathrm{T}}N \quad \text{(A25)}$$

Proof. Define $Y = \left(M^{\mathrm{T}}ZM - \varepsilon I_n\right)^{-\frac{1}{2}} M^{\mathrm{T}}ZA + \left(M^{\mathrm{T}}ZM - \varepsilon I_n\right)^{\frac{1}{2}} FN$. Considering $Y^{\mathrm{T}}Y \geq 0$, we can obtain:

$$A^{\mathrm{T}}ZMFN + N^{\mathrm{T}}F^{\mathrm{T}}M^{\mathrm{T}}ZA + N^{\mathrm{T}}F^{\mathrm{T}}\left(M^{\mathrm{T}}ZM - \varepsilon I_n\right)FN + A^{\mathrm{T}}ZM\left(M^{\mathrm{T}}ZM - \varepsilon I_n\right)^{-1}M^{\mathrm{T}}ZA \geq 0 \quad \text{(A26)}$$

Rearranging some terms, the lemma is proved. □

Now, using Lemma 3:

$$-(\varphi_2)^{\mathrm{T}} ZMF(k) J_1 - J_1^{\mathrm{T}} F^{\mathrm{T}}(k) M^{\mathrm{T}} Z\varphi_2 - J_1^{\mathrm{T}} F^{\mathrm{T}}(k) M^{\mathrm{T}} ZMF(k) J_1 \leq (\varphi_2)^{\mathrm{T}} ZM\left(M^{\mathrm{T}} ZM - \varepsilon_1\right)^{-1} M^{\mathrm{T}} Z\varphi_2 - \varepsilon_1 J_1^{\mathrm{T}} J_1. \quad (A27)$$

Thus,

$$\delta\omega_1 \leq \frac{2}{d-1}((\varphi_2)^{\mathrm{T}} ZM\left(M^{\mathrm{T}} ZM - \varepsilon_1\right)^{-1} M^{\mathrm{T}} Z\varphi_2 - \varepsilon_1 J_1^{\mathrm{T}} J_1 - \varphi_2^{\mathrm{T}} Z\varphi_2). \quad (A28)$$

Similarly,

$$\delta\omega_2 \leq \frac{6(d-2)}{d(d-1)}\left\{(\Pi_2)^{\mathrm{T}} ZM\left(M^{\mathrm{T}} ZM - \varepsilon_2\right)^{-1} M^{\mathrm{T}} Z\Pi_2 - \left(\frac{d}{d-2}\right)^2 \varepsilon_2 J_1^{\mathrm{T}} J_1 - \Pi_2^{\mathrm{T}} Z\Pi_2\right\} \quad (A29)$$

$$\delta\omega_3 \leq \left(L_1^2\right)^{\mathrm{T}} QM\left(M^{\mathrm{T}} QM - \varepsilon_3\right)^{-1} M^{\mathrm{T}} QL_1^2 - \varepsilon_3 J_1^{\mathrm{T}} J_1 - \left(L_1^2\right)^{\mathrm{T}} QL_1^2 \quad (A30)$$

$$\delta\omega_4 \leq \left(L_1^2 - L_0^2\right)^{\mathrm{T}} ZM\left(M^{\mathrm{T}} ZM - \varepsilon_4\right)^{-1} \times M^{\mathrm{T}} Z\left(L_1^2 - L_0^2\right)^{\mathrm{T}} - \varepsilon_4 J_1^{\mathrm{T}} J_1 - \left(L_1^2 - L_0^2\right)^{\mathrm{T}} Z\left(L_1^2 - L_0^2\right) \quad (A31)$$

where Π_2, L_1^2, L_0^2, φ_2 and J_1 are defined in (35h). Finally

$$\delta\omega \leq \bar{\omega} \quad (A32)$$

where $\bar{\omega}$ is defined in (35c).

Appendix B

In order to simplify the representation, the state vector of the system (1) is demonstrated as follows:

$$x(k) = e_1\xi(x) = L_0^1\xi(x) \quad (B1)$$

where $\xi(x)$ and e_1 are defined in (5a) and (5b), respectively. Then the one-step ahead state is:

$$x(k+1) = Ax(k) + A_d x(k-d) = Ae_1 + A_d e_2 = L_1^1\xi(x). \quad (B2)$$

Similarly, the m-step ahead state $x(k+m)$ can be represented as

$$x(k+m) = Ax(k+m-1) + A_d x(k+m-d) = L_m^1\xi(x) \quad (B3)$$

using model (1) repeatedly. L_j^1 in the above equations are defined in Remark 2.

References

[1] T. Shi, H. Y. Su, J. Chu. On stability and stabilisation for uncertain stochastic systems with time-delay and actuator saturation. *International Journal of Systems Science*, vol. 41, no. 5, pp. 501–509, 2010. DOI: 10.1080/00207720903072282.

[2] A. G. Wu, G. R. Duan. On delay-independent stability criteria for linear time-delay systems. *International Journal of Automation and Computing*, vol. 4, no. 1, pp. 95–100, 2007. DOI: 10.1007/s11633-007-0095-3.

[3] C. K. Zhang, Y. He, L. Jiang, M. Wu, H. B. Zeng. Summation inequalities to bounded real lemmas of discrete-time systems with time-varying delay. *IEEE Transactions on Automatic Control*, vol. 62, no. 5, pp. 2582–2588, 2017. DOI: 10.1109/TAC.2016.2600024.

[4] Y. He, M. Wu, G. P. Liu, J. H. She. Output feedback stabilization for a discrete-time system with a time-varying delay. *IEEE Transactions on Automatic Control*, vol. 53, no. 10, pp. 2372–2377, 2008. DOI: 10.1109/TAC.2008.2007522.

[5] H. Y. Shao, Q. L. Han. New stability criteria for linear discrete-time systems with interval-like time-varying delays. *IEEE Transactions on Automatic Control*, vol. 56, no. 3, pp. 619–625, 2011. DOI: 10.1109/TAC.2010.2095591.

[6] P. T. Nam, P. N. Pathirana, H. Trinh. Discrete Wirtinger-based inequality and its application. *Journal of the Franklin Institute*, vol. 352, no. 5, pp. 1893–1905, 2015. DOI: 10.1016/j.jfranklin.2015.02.004.

[7] A. Seuret, F. Gouaisbaut, E. Fridman. Stability of discrete-time systems with time-varying delays via a novel summation inequality. *IEEE Transactions on Automatic Control*, vol. 60, no. 10, pp. 2740–2745, 2015. DOI: 10.1109/TAC.2015.2398885.

[8] C. K. Zhang, Y. He, L. Jiang, M. Wu. An improved summation inequality to discrete-time systems with time-varying delay. *Automatica*, vol. 74, pp. 10–15, 2016. DOI: 10.1016/j.automatica.2016.07.040.

[9] X. M. Zhang, Q. L. Han. Abel lemma-based finite-sum inequality and its application to stability analysis for linear discrete time-delay systems. *Automatica*, vol. 57, pp. 199–202, 2015. DOI: 10.1016/j.automatica.2015.04.019.

[10] D. Aeyels, J. Peuteman. A new asymptotic stability criterion for nonlinear time-variant differential equations. *IEEE Transactions on Automatic Control*, vol. 43, no. 7, pp. 968–971, 1998. DOI: 10.1109/9.701102.

[11] A. Kruszewski, R. Wang, T. M. Guerra. Nonquadratic stabilization conditions for a class of uncertain nonlinear discrete time TS fuzzy models: A new approach. *IEEE Transactions on Automatic Control*, vol. 53, no. 2, pp. 606–611, 2008. DOI: 10.1109/TAC.2007.914278.

[12] A. A. Ahmadi, P. A. Parrilo. Non-monotonic Lyapunov functions for stability of discrete time nonlinear and switched systems. In *Proceedings of the 47th IEEE Conference on Decision and Control*, IEEE, Cancun, Mexico, pp. 614-621, 2008. DOI: 10.1109/CDC.2008.4739402.

[13] S. F. Derakhshan, A. Fatehi, M. G. Sharabiany. Nonmonotonic observer-based fuzzy controller designs for discrete time T-S fuzzy systems via LMI. *IEEE Transactions on Cybernetics*, vol. 44, no. 12, pp. 2557–2567, 2014. DOI: 10.1109/TCYB.2014.2310591.

[14] Y. Xie, J. W. Wen, L. Peng, H. Wang. Robust H_∞ control of average dwell time switching systems via non-monotonic Lyapunov function approach. In *Proceedings of the 3rd IEEE International Conference on Control Science and*

Systems Engineering, IEEE, Beijing, China, pp. 16-21, 2017. DOI: 10.1109/CCSSE.2017.8087886.

[15] Y. Xie, J. W. Wen, S. K. Nguang, L. Peng. Stability, l_2-gain, and robust H_∞ control for switched systems via N-step-ahead Lyapunov function approach. *IEEE Access*, vol. 5, pp. 26400-26408, 2017. DOI: 10.1109/ACCESS.2017.2771455.

[16] Y. J. Chen, M. Tanaka, K. Inoue, H. Ohtake, K. Tanaka, T. M. Guerra, A. Kruszewski, H. O. Wang. A nonmonotonically decreasing relaxation approach of Lyapunov functions to guaranteed cost control for discrete fuzzy systems. *IET Control Theory & Applications*, vol. 8, no. 16, pp. 1716–1722, 2014. DOI: 10.1049/iet-cta.2013.1132.

[17] A. Nasiri, S. K. Nguang, A. Swain, D. J. Almakhles. Robust output feedback controller design of discrete-time Takagi-Sugeno fuzzy systems: A non-monotonic Lyapunov approach. *IET Control Theory & Applications*, vol. 10, no. 5, pp. 545–553, 2016. DOI: 10.1049/iet-cta.2015.0750.

[18] A. Nasiri, S. K. Nguang, A. Swain, D. J. Almakhles. Reducing conservatism in an H_∞ robust state-feedback control design of T-S fuzzy systems: A nonmonotonic approach. *IEEE Transactions on Fuzzy Systems*, vol. 26, no. 1, pp. 386–390, 2018. DOI: 10.1109/TFUZZ.2017.2649580.

[19] J. W. Wen, S. K. Nguang, P. Shi, A. Nasiri. Robust H_∞ control of discrete-time nonhomogenous markovian jump systems via multistep Lyapunov function approach. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 47, no. 3, pp. 1439–1450, 2017. DOI: 10.1109/TSMC.2016.2617621.

[20] J. J. Hui, H. X. Zhang, X. Y. Kong, X. Zhou. On improved delay-dependent robust stability criteria for uncertain systems with interval time-varying delay. *International Journal of Automation and Computing*, vol. 12, no. 1, pp. 102–108, 2015. DOI: 10.1007/s11633-014-0822-5.

[21] S. Y. Xu, J. Lam, Y. Zou. Improved conditions for delay-dependent robust stability and stabilization of uncertain discrete time-delay systems. *Asian Journal of Control*, vol. 7, no. 3, pp. 344–348, 2005. DOI: 10.1111/j.1934-6093.2005.tb00244.x.

[22] V. K. R. Kandanvli, H. Kar. Robust stability of discrete-time state-delayed systems employing generalized overflow nonlinearities. *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 9, pp. 2780–2787, 2008. DOI: 10.1016/j.na.2007.08.050.

[23] B. Shafai, H. Sadaka, R. Ghadami. Robust stability and constrained stabilization of discrete-time delay systems. *IFAC Proceedings Volumes*, vol. 45, no. 14, pp. 31–36, 2012. DOI: 10.3182/20120622-3-US-4021.00058.

[24] K. Ramakrishnan, G. Ray. Robust stability criteria for a class of uncertain discrete-time systems with time-varying delay. *Applied Mathematical Modelling*, vol. 37, no. 3, pp. 1468–1479, 2013. DOI: 10.1016/j.apm.2012.03.045.

[25] L. Li, F. C. Liao. Robust preview control for a class of uncertain discrete-time systems with time-varying delay. *ISA Transactions*, vol. 73, pp. 11–21, 2018. DOI: 10.1016/j.isatra.2018.01.005.

[26] C. C. Sun. Stabilization for a class of discrete-time switched large-scale systems with parameter uncertainties. *International Journal of Automation and Computing*, vol. 16, no. 4, pp. 543–552, 2019. DOI: 10.1007/s11633-016-0966-6.

[27] P. P. Khargonekar, I. R. Petersen, K. Zhou. Robust stabilization of uncertain linear systems: Quadratic stabilizability and H_∞ infinity/control theory. *IEEE Transactions on Automatic Control*, vol. 35, no. 3, pp. 356–361, 1990. DOI: 10.1109/9.50357.

[28] M. Wu, Y. He, J. H. She. *Stability Analysis and Robust Control of Time-delay Systems*, Berlin, Germany: Springer, 2010. DOI: 10.1007/978-3-642-03037-6.

[29] J. Lofberg. YALMIP: A toolbox for modeling and optimization in MATLAB. In *Proceedings of 2004 IEEE International Conference on Robotics and Automation*, New Orleans, USA, pp. 284-289, 2004. DOI: 10.1109/CACSD.2004.1393890.

[30] S. Y. Xu, J. Lam, B. Y. Zhang, Y. Zou. A new result on the delay-dependent stability of discrete systems with time-varying delays. *International Journal of Robust and Nonlinear Control*, vol. 24, no. 16, pp. 2512–2521, 2014. DOI: 10.1002/rnc.3006.

[31] J. Yoneyama, T. Tsuchiya. New delay-dependent conditions on robust stability and stabilisation for discrete-time systems with time-delay. *International Journal of Systems Science*, vol. 39, no. 10, pp. 1033–1040, 2008. DOI: 10.1080/00207720802088116.

[32] C. Peng. Improved delay-dependent stabilisation criteria for discrete systems with a new finite sum inequality. *IET Control Theory & Applications*, vol. 6, no. 3, pp. 448–453, 2012. DOI: 10.1049/iet-cta.2011.0109.

[33] O. M. Kwon, M. J. Park, J. H. Park, S. M. Lee, E. J. Cha. Improved robust stability criteria for uncertain discrete-time systems with interval time-varying delays via new zero equalities. *IET Control Theory & Applications*, vol. 6, no. 16, pp. 2567–2575, 2012. DOI: 10.1049/iet-cta.2012.0257.

[34] G. Baruah, S. Majhi, C. Mahanta. Design of FOPI controller for time delay systems and its experimental validation. *International Journal of Automation and Computing*, vol. 16, no. 3, pp. 310–328, 2019. DOI: 10.1007/s11633-018-1165-4.

[35] M. Sadeghassadi, A. Fatehi, A. K. Sedigh, S. Hosseini. On the structural optimization of a neural network model predictive controller. *Industrial & Engineering Chemistry Research*, vol. 52, no. 1, pp. 394–407, 2012. DOI: 10.1021/ie201815r.

[36] D. Shaghaghi, A. Fatehi, A. Khaki-Sedigh. Multi-linear model set design based on the nonlinearity measure and H-gap metric. *ISA Transactions*, vol. 68, pp. 1–13, 2017. DOI: 10.1016/j.isatra.2017.01.021.

[37] K. J. Åström, T. Hägglund. *PID Controllers: Theory, Design, and Tuning*, 2nd ed., Research Triangle Park, North Carolina, USA: Instrument Society of America, 1995.

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