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An Adaptive Full Order Sliding Mode Controller for Mismatched Uncertain Systems

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Abstract: In this paper, an adaptive full order sliding mode (FOSM) controller is proposed for strict feedback nonlinear systems with mismatched uncertainties. The design objective of the controller is to track a specified trajectory in presence of significant mismatched uncertainties. In the first step the dynamic model for the first state is considered by the desired tracking signal. After the first step the desired dynamic model for each state is defined by the previous one. An adaptive tuning law is developed for the FOSM controller to deal with the bounded system uncertainty. The major advantages offered by this adaptive FOSM controller are that advanced knowledge about the upper bound of the system uncertainties is not a necessary requirement and the proposed method is an effective solution for the chattering elimination from the control signal. The controller is designed considering the full-order sliding surface. System robustness and the stability of the controller are proved by using the Lyapunov technique. A systematic adaptive step by step design method using the full order sliding surface for mismatched nonlinear systems is presented. Simulation results validate the effectiveness of the proposed control law.

Keywords: Full order sliding mode, adaptive sliding mode, finite time convergence, reference tracking, mismatched uncertainty.

1 Introduction

Sliding mode (SM) control is widely used because of its simplicity in design, and particularly due to its robustness towards the disturbances and the plant uncertainties. When the system reaches the sliding mode, it is insensitive towards the matched external disturbances and the variations of the plant parameters. Thus the sliding mode control technique has been successfully applied to a large variety of practical systems such as robot manipulators, flexible space structures, underwater vehicles, aircrafts, spacecrafts, power systems, electrical motors, automotive engines or so on. The study of the sliding mode control considering matched uncertainty is widely studied in the literature[1−5]. However uncertainties existing in many physical systems may not always satisfy the matching condition. When the uncertainties/disturbances appear in the input channel, it satisfies the matching condition and it is termed as matched uncertainty. When the uncertainties/disturbances affect the other channels, then they do not satisfy the matching condition, so it is termed as mismatched uncertainty[2]. It is quite natural for a system that it can be affected by both matched and mismatched or unmatched perturbations. Nevertheless, traditional sliding mode design methods[6] may fail to give stabilization in presence of mismatched uncertainties. To reduce the effect of mismatched uncertainties[7−11], different methods have been combined with sliding mode. A linear matrix inequities (LMI) based sliding surface design method is developed in [9] for guaranteeing the quadratic stability. Similarly the effect of mismatched norm bounded uncertainties in the state matrix as well as the input matrix using LMI technique are studied in [12].

One significant research finding is that the stability of the system is guaranteed if the system trajectory is driven to a bounded region. Hence when the system contains mismatched perturbations, the information about the upper bound of perturbations is needed in order to achieve asymptotic stability[9]. In [11] and [13, 14], the sliding mode controllers are designed to stabilize a given system but the main drawback is, the controllers are applicable to a certain kind of mismatched uncertain system, satisfying some necessary conditions. For linear systems a technique was developed in [15], where asymptotic stability could be achieved without requiring the information about the upper bound of the system′s model uncertainties. Here an adaptive mechanism was embedded in the controller as well as in the sliding surface, but the control input obtained by using the above-mentioned method was not smooth and high frequency chattering was present which made the algorithm substantially difficult to apply in practice. In [16-18] disturbance observers are used to control a higher order uncertain systems with mismatched uncertainties. Controlling of a class of nonholonomic systems affected by uncertainties using backstepping based second order sliding mode control is presented in [19]. In [20] backstepping design is combined with sliding mode control for systems in strict-feedback form with parameter uncertainties and the same is extended to the multi-input case in [21]. In [22] at each step a virtual control is designed using backstepping

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technique and finally at the end the system is represented as an auxiliary second order subsystem and a second order sliding mode control is applied. For the strict feedback uncertain nonlinear systems, backstepping[22] approach in a step-by-step design method is very useful to attenuate the mismatched perturbations.

Algorithms using the quasi continuous higher order sliding mode (HOSM) for strict-feedback systems are presented in [23, 24]. The control technique achieves finite-time tracking of the desired output in the presence of smooth mismatched uncertainties. Similarly the stabilization of strict feedback systems with mismatched uncertainties using second-order sliding mode is presented in [25].

Roughly speaking, all the aforementioned sliding mode design methods can be divided into two categories. The first category mainly focuses on the stability of linear systems under mismatched uncertainties via some classical control design tools, such as the LMI technique[9, 12], adaptive approach[14, 25, 26]. Secondly the quasi continuous higher order[23, 24] and second order sliding mode algorithms[25] are for nonlinear strict-feedback systems.

Note that the above two categories of sliding mode methods handle the mismatched uncertainties in a robust way, the second category handles a much more generalized system (strict-feedback) is considered, but the above algorithms are developed with the assumption that the knowledge of the upper bounds of the system uncertainty must be known. But in case of real time system the upper bound of the uncertainty may not be always available. The advantage of the proposed technique is to adaptively estimate the bounds of the uncertainty without having any prior knowledge about it. Another drawback is that the control laws proposed in [23, 24] are not smooth, and they suffers substantial chattering. The proposed technique is more efficient in reducing the chattering from the control input. The finite time convergence is also guaranteed by the proposed step-by-step adaptive full order sliding mode (FOSM) controller using a full order sliding surface. In conventional sliding mode, the sliding surface is chosen so that it has desirable reduced-order dynamics when the system is in the ideal sliding-mode. In this paper, a full-order sliding-mode manifold is utilized, and a step by step control is proposed. During the ideal sliding-mode, the system behaves with a desirable full-order dynamics, and not with reduced-order dynamics[27].

The aim of this paper is to present a step by step design of an adaptive FOSM controller for nonlinear strict feedback systems with mismatched uncertainties for which.

1) The establishment of HOSM ensures finite time convergence of the system′s tracking errors in finite time.

2) The robustness and the finite time tracking of the output are ensured in spite of the presence of unmatched perturbations, i.e., parameter uncertainties and external disturbances.

3) The adaptive tuning law guarantees the estimation of system uncertainties whose upper bounds are not required to be known in advance.

4) The use of FOSM scheme is more successful and efficient in reducing chattering as compared to [23, 24].

The design procedure is carried out by considering each state as a separate plant. Then for each state a sliding surface is designed using FOSM technique. The first step is defined by the desired tracking signal by considering a virtual control. After the first step, the desired dynamics for each state is defined by the previous one and similarly the sliding surface and virtual control law are obtained by using FOSM technique and at the last step final control law is developed.

The control law at each step consists of two parts, one is equivalent or nominal control and the other one is switching control. The equivalent control law compensates the nominal part of the system and the switching control law compensates the effects of perturbations to achieve the desired control objectives.

In this paper, a homogeneity based adaptive FOSM controller is proposed for strict-feedback systems with mismatched uncertainties. As compared to [23–25], the adaptive technique guarantees the estimation of uncertainties without knowing their upper bound a priori[28−40]. The gain dynamics also ensure that there is no over-estimation of the gain with respect to the unknown value of uncertainties.

The occurrence of HOSM is proved based on homogeneity method. The implementation of this algorithm needs the calculation of the sliding function and its first and higher order derivatives. For this reason, robust exact differentiators[41] are employed. A systematic design method of step by step adaptive FOSM control for a mismatched nonlinear strict-feedback systems is presented.

The rest of the paper is organized as follows. Section 2 explains the class of systems and the problem formulation. Section 3 discusses the design procedure of the proposed step by step adaptive FOSM control. Effectiveness of the proposed controller is demonstrated in Section 4 by performing simulation studies. The conclusions are drawn in Section 5.

2 Problem formulation

Consider a nonlinear system given by[23]

$$\begin{aligned}
&\dot{x}_1 = f_1(x_1,t) + g_1(x_1,t)x_2 + d_1(x_1,t)\\
&\dot{x}_i = f_i(\bar{x}_i,t) + g_i(\bar{x}_i,t)x_{i+1} + d_i(\bar{x}_i,t),\quad 2 \leq i < n\\
&\dot{x}_n = f_n(x,t) + g_n(x,t)u + d_n(x,t)\\
&y = x_1
\end{aligned}\tag{1}$$

where n is the order of the system, $x = [x_1\ x_2 \cdots x_n]^{\rm T} \in \mathbf{R}^n$ represents measurable state vector and $\bar{x}_i = [x_1\ x_2 \cdots x_i]^{\rm T}$. $f_i(\bar{x}_i,t)$ and $g_i(\bar{x}_i,t) \neq 0$ are smooth nonlinear functions available for feedback. $u \in \mathbf{R}$ is the control input. $y = x_1$ is the output. The term $d_i(\bar{x}_i,t)$ represents parametric uncertainties and external disturbances.

Assumption 1. The disturbances $d_i(\bar{x}_i, t)$ are assumed to be smooth and bounded, and differentiable.

The problem of interest is to design a controller such that the output $y = x_1$ tracks a desired smooth reference x_d in spite of the presence of the unknown bounded perturbations $d_i(\bar{x}_i, t)$. For further development, the tracking error is considered as $\sigma_1 = x_1 - x_d$.

Consider a nonlinear system given by

$$\begin{aligned} \dot{\xi} &= a(t,\xi) + b(t,\xi)u \\ \sigma &= \sigma(t,\xi) \end{aligned} \tag{2}$$

where $\xi \in \mathbf{R}^n, u \in \mathbf{R}, \sigma \in \mathbf{R}$, a, b and σ are smooth functions assumed to be unknown.

Assumption 2. The relative degree (p) of the system (2) with respect to the output is assumed to be constant and known a priori.

Remark 1. For sake of clarity, only the case of $p = n$ is considered, however the approach can be extended for the case of $p < n$ also, if the zero dynamics stability is assumed[24].

For r-th order sliding mode of the system (2), the r-th derivative of $\sigma(t, \xi)$ satisfies the equation

$$\sigma^{(r)} = \aleph(t,\xi) + \hbar(t,\xi)u \tag{3}$$

where $\hbar(t,\xi) \neq 0$ holds and $\aleph(t,\xi) = \sigma^{(r)}|_{u=0}, \hbar(t,\xi) = (\frac{\partial}{\partial u})\sigma^{(r)}$. The uncertainty does not allow the immediate reduction of (2) to (3).

Assumption 3. The following inequalities hold for some values of $K_m, K_M > 0$ and $C > 0$, which can be expressed as

$$0 < K_m \leq \frac{\partial}{\partial u}\sigma^{(r)} \leq K_M, \quad |\sigma^{(r)}|_{u=0} \leq C. \tag{4}$$

Then (3), (4) imply the differential inclusion

$$\sigma^{(r)} \in [-C, C] + [K_m, K_M]u. \tag{5}$$

The above problem can be solved by designing a full order sliding surface along with a FOSM controller, in such a way that $\sigma = \dot{\sigma} = \cdots = \sigma^{(r-1)} = 0$ is achieved in finite time and also the output $y = x_1$ tracks a smooth desired reference x_d in spite of the presence of unknown bounded mismatched perturbations.

3 Step by step FOSM controller design

Consider the system given by (1), where each state can be seen as a function of the previous states. If we assume that $x_{i+1}(t) = \phi_i, \ \forall i = 1, \cdots, n-1$, i.e., as a virtual control then for the first state equation of (1), i.e., $\dot{x}_1 = f_1(x_1, t) + g_1(x_1, t)x_2 + d_1(x_1, t)$, here x_2 can be seen as virtual control and then $d_1(x_1, t)$ is the matched uncertainty as it is satisfying the matching condition with the virtual control.

Now the algorithm is developed in such a way that the output $y = x_1$ tracks the reference signal x_d.

The control block diagram of the proposed step by step method can be shown in Fig. 1.

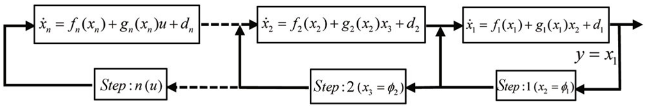

Fig. 1 Block diagram

Step 1. Since the system is of n-th order. Thus the system consists of n nested control loops. The first state equation of (1) is cascaded with other $n-1$ state equations. For the inner-loop, let x_2 correspond to a virtual control input (ϕ_1) acting on the dynamics of first state equation of (1) with the aim of steering the output $y = x_1$ to desired reference signal x_d in finite time. To achieve this desired control perspective, an n-th order full sliding surface will be designed. Now the tracking error is given by $\sigma_1 = x_1 - x_d$ and also assume that $(n-1)$-th derivatives are available for feedback. Thus a full order sliding surface[27] can be obtained as

$$\begin{aligned} s_1 = \ & \dot{\sigma}_1 + k_1|\sigma_1|^{\alpha_1}\mathrm{sgn}(\sigma_1) + k_2|\dot{\sigma}_1|^{\alpha_2}\mathrm{sgn}(\dot{\sigma}_1) + \cdots + \\ & k_n|\sigma_1^{(n-1)}|^{\alpha_n}\mathrm{sgn}(\sigma_1^{(n-1)}) \end{aligned} \tag{6}$$

where $k_1, k_2, \cdots, k_n$ are the positive constants, chosen such that the polynomial

$$P(\psi) = \psi^n + k_n\psi^{n-1} + \cdots + k_2\psi + k_1 \tag{7}$$

is Hurwitz.

The exponents $\alpha_1, \alpha_2, \cdots, \alpha_n$ can be obtained by satisfying the following conditions[42, 43]

$$\begin{aligned} \alpha_1 &= \alpha, \quad n = 1 \\ \alpha_{i-1} &= \tfrac{\alpha_i \alpha_{i+1}}{2\alpha_{i+1} - \alpha_i}, \quad i = 2, \cdots, n, \quad \forall n \geq 2 \end{aligned} \tag{8}$$

where $\alpha_{n+1} = 1$, $\alpha_n = \alpha, \alpha \in (1-\epsilon, 1), \epsilon \in (0, 1)$.

When the ideal sliding-mode $s_1 = 0$ is reached, the motion can be expressed as

$$\begin{aligned} & \dot{\sigma}_1 + k_1|\sigma_1|^{\alpha_1}\mathrm{sgn}(\sigma_1) + k_2|\dot{\sigma}_1|^{\alpha_2}\mathrm{sgn}(\dot{\sigma}_1) + \cdots + \\ & \qquad k_n|\sigma_1^{(n-1)}|^{\alpha_n}\mathrm{sgn}(\sigma_1^{(n-1)}) = 0 \end{aligned} \tag{9}$$

if $k_1, k_2, \cdots, k_n$ are properly chosen such as mentioned above and the exponents $\alpha_1, \alpha_2, \cdots, \alpha_n$ satisfy the conditions (7) and (8), then $\sigma_1, \cdots, \sigma_1^{(n-1)}$ converges to the equilibrium from any initial condition. For detailed analysis, the readers are referred to the work by Bhat and Bernstein[42] and Feng et al.[27]

Remark 2. The missing derivatives of $\sigma_i, i = 1, \cdots, n-1$ can be estimated by means of the robust exact finite time convergent differentiator[41].

The n-th order full sliding surface s_1 will converge to zero and $\sigma_1 = x_1 - x_d$ will vanish in finite time if the virtual control law (ϕ_1) is chosen as

$$\phi_1 = g_1(x_1,t)^{-1}[u_{eq1} + u_1] \tag{10}$$

$$\begin{aligned} u_{eq1} = &- f_1(x_1,t) - k_1|\sigma_1|^{\alpha_1}\mathrm{sgn}(\sigma_1) - \\ & k_2|\dot\sigma_1|^{\alpha_2}\mathrm{sgn}(\dot\sigma_1) - \cdots - \\ & k_n|\sigma_1^{(n-1)}|^{\alpha_n}\mathrm{sgn}(\sigma_1^{(n-1)}) + \dot x_d \end{aligned} \tag{11}$$

$$\dot u_1 + \tau_1 u_1 = v_1 \tag{12}$$

$$v_1 = - k_{v1}\mathrm{sgn}(s_1) \tag{13}$$

where $u_1(0) = 0$, $k_{v1} = (k_{d1} + k_{\tau_1} + \eta_1)$ and k_{d1}, k_{τ_1}, η_1 are the positive constants. $\tau_1 > 0$ and $k_{\tau_1} \geq |\tau_1 u_1|$.

If the disturbance and its derivative are bounded by certain values $|\dot d_1(x,t)| \leq k_{d1}$ then the control signal u_1 is bounded and the following inequality holds $|\tau_1 u_1| \leq k_{\tau_1}$.

The value of k_{v1} is chosen as $k_{v1} = k_{d1} + k_{\tau_1} + \eta_1$, which means that system (1) will reach to the equilibrium $s_1 = 0$ in finite time. In practice, the upper bound of the system uncertainties are often unknown in advance and hence difficult to find. Thus an adaptive tuning law can be adopted to estimate k_{v1}[34, 37, 44].

The i-th sliding surface s_1 will reach the equilibrium in finite time if the tuning function is chosen as $\dot{\hat k}_{v1} = \gamma_1|s_1|$, where $\hat k_{v1}$ is the estimate value of k_{v1}.

Step *i*. Now the i-th step design objective will be to develop an i-th order FOSM controller such that $x_i = \phi_{i-1}$ is achieved. To achieve the desired performance i-th full order sliding surface is chosen as

$$\begin{aligned} s_i = \ & \dot\sigma_i + k_1|\sigma_i|^{\alpha_1}\mathrm{sgn}(\sigma_i) + k_2|\dot\sigma_i|^{\alpha_2}\mathrm{sgn}(\dot\sigma_i) + \cdots + \\ & k_i|\sigma_i^{(i-1)}|^{\alpha_i}\mathrm{sgn}(\sigma_i^{(i-1)}) \end{aligned} \tag{14}$$

where $k_1, k_2, \cdots, k_i$ and $\alpha_1, \alpha_2, \cdots, \alpha_i$ are design constants which can be obtained from (7) and (8). The i-th sliding surface s_i will converge to the equilibrium and $\sigma_i = x_i - \phi_{i-1}$ will vanish in finite time if the virtual control law (ϕ_i) is chosen as

$$\phi_i = g_i(\bar x_i,t)^{-1}[u_{eqi} + u_i] \tag{15}$$

$$\begin{aligned} u_{eqi} = \ & - f_i(\bar x_i,t) - k_1|\sigma_i|^{\alpha_1}\mathrm{sgn}(\sigma_i) - \\ & k_2|\dot\sigma_i|^{\alpha_2}\mathrm{sgn}(\dot\sigma_i) - \cdots - \\ & k_i|\sigma_i^{(i-1)}|^{\alpha_i}\mathrm{sgn}(\sigma_i^{(i-1)}) + \dot\phi_{i-1} \end{aligned} \tag{16}$$

$$\dot u_i + \tau_i u_i = v_i \tag{17}$$

$$v_i = -k_{vi}\mathrm{sgn}(s_i) \tag{18}$$

where $u_i(0) = 0$, $k_{vi} = (k_{di} + k_{\tau_i} + \eta_i)$ and $k_{di}, k_{\tau_i}, \eta_i$ are positive constants. $\tau_i > 0$ and $k_{\tau_i} \geq |\tau_i u_i|$.

If $|\dot d_i(\bar x_i,t)| \leq k_{di}$ and also the the control signal u_i satisfies the inequality $|\tau_i u_i| \leq k_{\tau_i}$, then with $k_{vi} = k_{di} + k_{\tau_i} + \eta_i$, error state σ_i will reach the equilibrium in finite time.

The value of k_{vi} can be estimated adaptively by $\hat k_{vi}$, if the following tuning function is chosen $\dot{\hat k}_{vi} = \gamma_i|s_i|$, where $\gamma_i > 0$.

Step *n*. Now at the last step the relative degree of the system with respect to the control input is 1. Thus the 1st order sliding mode will be designed such that $\sigma_n = x_n - \phi_{n-1}$ converges to zero. To vanish the $\sigma_n \to 0$ in finite time, let us choose a sliding surface

$$s_n = \dot\sigma_n + k_n|\sigma_n|^{\alpha_1}\mathrm{sgn}(\sigma_n) \tag{19}$$

where $k_n > 0$ and $\alpha_1 \in (1 - \epsilon_1, 1)$, $\epsilon_1 \in (0,1)$. The last sliding surface s_n will converge to the equilibrium and $\sigma_n = x_n - \phi_{n-1}$ will vanish in finite time if the actual control law (u) is chosen as

$$u = g_n(x,t)^{-1}[u_{eqn} + u_n] \tag{20}$$

$$u_{eqn} = -f_n(x,t) - k_n|\sigma_n|^{\alpha_1}\mathrm{sgn}(\sigma_n) + \dot\phi_{n-1} \tag{21}$$

$$\dot u_n + \tau_n u_n = v_n \tag{22}$$

$$v_n = -k_{vn}\mathrm{sgn}(s_n) \tag{23}$$

where $u_n(0) = 0$, $k_{vn} = (k_{dn} + k_{\tau_n} + \eta_n)$ and $k_{dn}, k_{\tau_n}, \eta_n$ are positive constants. $\tau_n > 0$ and $k_{\tau_n} \leq |\tau_n u_n|$.

If $|\dot d_n(x,t)| \leq k_{dn}$ and also if the control signal u_n satisfies $|\tau_n u_n| \leq k_{\tau_n}$, then with the value of $k_{vn} = k_{dn} + k_{\tau_n} + \eta_n$, sliding surface s_n will reach the equilibrium in finite time. The value of k_{vn} can be estimated adaptively if the following tuning function is chosen $\dot{\hat k}_{vn} = \gamma_n|s_n|$, where $\gamma_n > 0$.

Remark 3. It is quite clear that the derivative of $\phi_i = f(\sigma_i)$, will be extremely large or infinite as $\sigma_i = 0$ and $\dot\sigma_i \neq 0$ while calculating the virtual and final control law. To avoid this problem, the following modifications can be used

$$\frac{\mathrm{d}}{\mathrm{d}t}[|\sigma_i|^{\alpha_i}\mathrm{sgn}(\sigma_i)] = \begin{cases} \alpha_i|\sigma_i|^{\alpha_i-1}\dot\sigma_i, & \sigma_i \neq 0, \dot\sigma_i \neq 0 \\ \alpha_i|\Delta_i|^{\alpha_i-1}\dot\sigma_i, & \sigma_i = 0, \dot\sigma_i \neq 0 \\ 0, & \dot\sigma_i = 0 \end{cases} \tag{24}$$

where $\alpha_i > 0$, and $\Delta_i > 0$ is a small positive constant. From the work by Zhao et al.[45], it can be shown that the above assumption can be made in order to avoid the singularity. For more detailed analysis, the work shown in [45] can be referred to. Similar procedure can be applied for the derivative calculation of other terms, such as $|\sigma_i^{(i-1)}|^{\alpha_i}\mathrm{sgn}(\sigma_i^{(i-1)})$.

Theorem 1. The system $\dot x_1 = f_1(x_1,t) + g_1(x_1,t)x_2 + d_1(x_1,t)$ will reach $s_1 = 0$ in finite time and will converge to zero along $s_1 = 0$ in finite time, if the virtual control law (ϕ_1) is chosen as follows:

$$\phi_1 = g_1(x_1,t)^{-1}[u_{eq1} + u_1] \tag{25}$$

$$\begin{aligned} u_{eq1} = \ & - f_1(x_1,t) - k_1|\sigma_1|^{\alpha_1}\mathrm{sgn}(\sigma_1) - \\ & k_2|\dot\sigma_1|^{\alpha_2}\mathrm{sgn}(\dot\sigma_1) - \cdots - \\ & k_n|\sigma_1^{(n-1)}|^{\alpha_n}\mathrm{sgn}(\sigma_1^{(n-1)}) + \dot x_d \end{aligned} \tag{26}$$

$$\dot u_1 + \tau_1 u_1 = v_1 \tag{27}$$

$$v_1 = -k_{v1}\mathrm{sgn}(s_1) \tag{28}$$

where $u_1(0) = 0$, $k_{v1} = (k_{d1} + k_{\tau_1} + \eta_1) > 0$, $\tau_1 > 0$ and $k_{\tau_1} \geq |\tau_1 u_1|$.

Proof. For system (1), full order sliding manifold s_1 can be rewritten as follows:

$$\begin{aligned}s_1 = &\dot{x}_1 + k_1|\sigma_1|^{\alpha_1}\operatorname{sgn}(\sigma_1) + k_2|\dot{\sigma}_1|^{\alpha_2}\operatorname{sgn}(\dot{\sigma}_1) + \cdots + \\ &k_n|\sigma_1^{(n-1)}|^{\alpha_n}\operatorname{sgn}(\sigma_1^{(n-1)}) - \dot{x}_d = \\ &f_1(x_1,t) + g_1(x_1,t)[u_{eq1} + u_1] + d_1(x_1,t) + \\ &k_1|\sigma_1|^{\alpha_1}\operatorname{sgn}(\sigma_1) + k_2|\dot{\sigma}_1|^{\alpha_2}\operatorname{sgn}(\dot{\sigma}_1) + \cdots + \\ &k_n|\sigma_1^{(n-1)}|^{\alpha_n}\operatorname{sgn}(\sigma_1^{(n-1)}) - \dot{x}_d\end{aligned} \tag{29}$$

where $d_1(x_1,t)$ is the uncertainty. Substituting the values of (25) into (29) we have

$$s_1 = d_1(x_1,t) + u_1. \tag{30}$$

Taking the derivative of (30) the above equation gives

$$\dot{s}_1 = \dot{d}_1(x_1,t) + \dot{u}_1. \tag{31}$$

Now consider the following Lyapunov function $V_1 = \frac{1}{2s_1^2}$. Taking the derivative of V_1

$$\begin{aligned}\dot{V}_1 = &s_1\dot{s}_1 = s_1[\dot{d}_1(x_1,t) + \dot{u}_1] = \\ &s_1[\dot{d}_1(x_1,t) + \dot{u}_1 + \tau_1 u_1 - \tau_1 u_1] = \\ &s_1[\dot{d}_1(x_1,t) + v_1 - \tau_1 u_1] \leq \\ &[\dot{d}_1(x_1,t)s_1 - k_{v1}s_1\operatorname{sgn}(s_1) - \tau_1 u_1 s_1] \leq \\ &[\dot{d}_1(x_1,t)s_1 - (k_{d_1} + k_{\tau_1} + \eta_1)|s_1| - \tau_1 u_1 s_1] \leq \\ &[(\dot{d}_1(x_1,t)s_1 - k_{d1}|s_1|) + (-\tau_1 u_1 s_1 - k_{\tau 1}|s_1|) - \\ &\eta_1|s_1|] \leq \\ &-\eta_1|s_1| \leq \\ &-\eta_1 V_1^{\frac{1}{2}}.\end{aligned} \tag{32}$$

As per the assumptions if the disturbance and its derivative are bounded by certain values $|\dot{d}_1(x_1,t)| \leq k_{d1}$ and also the control sgnal u_1 is bounded and the following inequality holds $|\tau_1 u_1| \leq k_{\tau_1}$.

The value of k_{v1} is chosen as $k_{v1} = k_{d1} + k_{\tau_1} + \eta_1$, which means that system (1) will reach to $s_1 = 0$ in finite time. □

Theorem 2. The overall stability, the convergence and the boundness of the states $x_1, \cdots, x_i, \cdots, x_n$ can be obtained if we consider the virtual control along with the actual control law as given in (10), (15) and (20).

Proof. The overall stability can be obtained by following way, for the second step a Lyapunov function is chosen as

$$V_2 = \underbrace{\frac{1}{2}s_1^2}_{V_1} + \frac{1}{2}s_2^2\ . \tag{33}$$

Its first derivative can be computed as

$$\begin{aligned}\dot{V}_2 \leq &-\eta_1 V_1^{\frac{1}{2}} - \eta_2|s_2| \leq \\ &-\eta_1 V_1^{\frac{1}{2}} - \eta_2(s_2^2)^{\frac{1}{2}} \leq \\ &-\tilde{\eta}_2[V_1 + s_2^2]^{\frac{1}{2}} \leq \\ &-\tilde{\eta}_2 V_2^{\frac{1}{2}}\end{aligned} \tag{34}$$

where $\tilde{\eta}_2 = \min\{\eta_1, \eta_2\}$.

The same procedure can be applied recursively to the remaining sliding surface variables. The Lyapunov function candidate for the last step can be obtained as

$$V_n = V_{n-1} + \frac{1}{2}s_n^2. \tag{35}$$

Using the virtual and actual control law (10), (15) and (20) one can easily obtain

$$\dot{V}_n \leq -\sum_{i=1}^{n}\eta_i|s_i| \leq -\tilde{\eta}_n V^{\frac{1}{2}} \tag{36}$$

where $\tilde{\eta}_n = \min[\eta_1, \cdots, \eta_i, \cdots, \eta_n]$.

Thus, it proves the overall stability of the system. Since s_i converges to zero in finite time and with the surface chosen as (6), (14) and (19) the convergence of σ_i is also guaranteed. For detailed explanation, the work by Feng et al.[27] can be referred to. □

Remark 4. In the above Theorem 2, the control signal (25) is equivalent to a low-pass filter, where v_1 is the input, u_n is the output of the filter, and τ_1 is the time constant or bandwidth of the filter. The use of filter reduces the chattering considerably but the response gets little slower. From the work of Feng et al.[27], it is quite clear that above consideration is quite useful for practical applications.

In practice, the upper bound of the system uncertainties are often unknown in advance and hence difficult to find. Thus an adaptive tuning law can be adopted to estimate k_{v1}[44].

Theorem 3. The system $\dot{x}_1 = f_1(x_1,t) + g_1(x_1,t)x_2 + d_1(x_1,t)$ will reach $s_1 = 0$ in finite time and will converge to zero along $s_1 = 0$ in finite time, if the virtual control law (ϕ_1) is chosen as

$$\phi_1 = g_1(x_1,t)^{-1}[u_{eq1} + u_1] \tag{37}$$

$$\begin{aligned}u_{eq1} = &-f_1(x_1,t) - k_1|\sigma_1|^{\alpha_1}\operatorname{sgn}(\sigma_1) - \\ &k_2|\dot{\sigma}_1|^{\alpha_2}\operatorname{sgn}(\dot{\sigma}_1) - \cdots - \\ &k_n|\sigma_1^{(n-1)}|^{\alpha_n}\operatorname{sgn}(\sigma_1^{(n-1)}) + \dot{x}_d\end{aligned} \tag{38}$$

$$\dot{u}_1 + \tau_1 u_1 = v_1 \tag{39}$$

$$v_1 = -k_{v1}\operatorname{sgn}(s_1) \tag{40}$$

with $\gamma_1 > 0$, the value of k_{v1} can be estimated adaptively if the following tuning function is chosen as[44]

$$\dot{\hat{k}}_{v1} = \gamma_1|s_1|. \tag{41}$$

Proof. Let us consider the following Lyapunov function[37, 46] $V_1 = \frac{1}{2}s_1^2 + \frac{1}{2\nu_1}\tilde{k}_{v1}^2$, where ν_1 is a positive constant, adaptation error $\tilde{k}_{v1} = \hat{k}_{v1} - k_{v1}$ and $\hat{k}_{v1}$ is the estimated value of the k_{v1}. Taking the derivative of Lyapunov function yields

$$\begin{aligned}\dot{V}_1 = &s_1\dot{s}_1 = s_1[\dot{d}_1(x_1,t) + \dot{u}_1] + \nu_1^{-1}\tilde{k}_{v1}\dot{\hat{k}}_{v1} \leq \\ &s_1[\dot{d}_1(x_1,t) - \hat{k}_{v1}\operatorname{sgn}(s_1) - \tau u_1] + \\ &\nu_1^{-1}(\hat{k}_{v1} - k_{v1})\dot{\hat{k}}_{v1}.\end{aligned}$$

If u_1 is bounded and using

$$\begin{aligned}\dot{\hat{k}}_{v1} =& \gamma_1|s_1| \le\\ & [|\dot{d}_1(x_1,t)||s_1| + k_{\tau_1}|s_1| - k_{v1}|s_1| + k_{v1}|s_1| -\\ & k_{v1}|s_1|] + \nu_1^{-1}\gamma_1(\hat{k}_{v1} - k_{v1})|s_1| \le\\ & [(|\dot{d}_1(x_1,t)||s_1| + k_{\tau_1} - k_{v1})|s_1| -\\ & (\hat{k}_{v1} - k_{v1})|s_1|] + \nu_1^{-1}\gamma_1(\hat{k}_{v1} - k_{v1})|s_1| \le\\ & [-\underbrace{(-|\dot{d}_1(x_1,t)| - k_{\tau_1} + k_{v1})}_{\beta_{s1}}|s_1| -\\ & (\hat{k}_{v1} - k_{v1})\underbrace{(|s_1| - \nu_1^{-1}\gamma_1|s_1|)}_{\beta_{a1}}] \le\\ & -\sqrt{2}\beta_{s1}\frac{|s_1|}{\sqrt{2}} - \sqrt{2\nu_1}\beta_{a1}\frac{1}{\sqrt{2\nu_1}}\tilde{k}_{v1} \le\\ & -\underbrace{\min\{\sqrt{2}\beta_{s1}, \sqrt{2\nu_1}\beta_{a1}\}}_{\beta_{x1}}(\frac{|s_1|}{\sqrt{2}} + \frac{1}{\sqrt{2\nu_1}}\tilde{k}_{v1}) \le\\ & -\beta_{x1}V_1^{\frac{1}{2}}.\end{aligned} \tag{42}$$

The above inequality holds if $\dot{\hat{k}}_{v1} = \gamma_1|s_1|$ and $\nu_1^{-1}\gamma_1 < 1$, also $k_{v1} = k_{d1} + k_{\tau_1} + \eta_1$, which yields $\beta_{s_1} > 0, \beta_{a1} > 0$. Now with $\beta_{x1} = \min\{\sqrt{2}\beta_{s1}, \sqrt{2\nu_1}\beta_{a1}\}$, which is a positive constant, hence the sliding surface converges to zero in finite time. □

In a similar way, stability and convergence of other sliding surfaces can also be proved.

Remark 5. By choosing a suitable adaptation tuning parameter γ_i one can effectively avoid high control activity in the reaching mode. When applying this tuning rule, the initial setting of the adaptation gain should be smaller than the upper bound of the system uncertainties. However, this is not really a restriction of the method since one can freely choose a sufficiently small number.

Remark 6. This tuning rule is applicable where $|s_i| = 0, \forall i = 1, \cdots, n$, is reachable. However, in real sliding mode control, $|s_i|$ cannot become exactly zero in finite time due to sampled computation, noisy measurements, nonlinearities and switching delays. Thus the adaptive parameter $\hat{k}_{vi}$ may increase boundlessly. A simple way of overcoming this difficulty is to modify the adaptive tuning law (41) by using the dead zone technique[30, 44, 47] as

$$\dot{\hat{k}}_{vi} = \begin{cases} \gamma_i|s_i|, & |s_i| \ge \varepsilon\\ 0, & |s_i| < \varepsilon\end{cases} \tag{43}$$

where ε is a small positive constant. The main feature of this approach is that it does not require a priori the knowledge of the uncertainty and disturbances. The increase of switching gain may be controlled by using a boundary layer neighboring the sliding surface. It means that accuracy has to be sacrificed in order to apply the previous controller and that the control gain is still over-estimated.

To reduce the over estimation of switching control gain the following tuning rule can also be used

$$\dot{\hat{k}}_{vi} = \rho_i(-\kappa_i\hat{k}_{vi} + |s_i|), \quad \forall i = 1, \cdots, n \tag{44}$$

where ρ_i, κ_i is a positive constant.

The convergence of s_1 can be proved in the following way: Let us consider the following Lyapunov function $V_1 = \frac{1}{2}s_1^2 + \frac{1}{2\rho_1}\tilde{k}_{v1}^2$. Adaptation error $\tilde{k}_{v1} = \hat{k}_{v1} - k_{v1}$ and $\hat{k}_{v1}$ estimates the value of k_{v1}. Taking the derivative of Lyapunov function yields

$$\begin{aligned}\dot{V}_1 =& s_1\dot{s}_1 = s_1[\dot{d}_1(x_1,t) + \dot{u}_1] + \rho_1^{-1}\tilde{k}_{v1}\dot{\hat{k}}_{v1} =\\ & s_1[\dot{d}_1(x_1,t) - \hat{k}_{v1}\mathrm{sgn}(s_1) - \tau_1 u_1] +\\ & \rho_1^{-1}(\hat{k}_{v1} - k_{v1})\dot{\hat{k}}_{v1}\end{aligned}$$

where $\hat{k}_{v1}$ estimates the value of k_{v1}.

If

$$|\tau_1 u_1| < k_{\tau_1}$$

then

$$\begin{aligned}\dot{\hat{k}}_{v1} =& \rho_1(-\kappa_1\hat{k}_{v1} + |s_1|) \le\\ & [|\dot{d}_1(x_1,t)||s_1| + |\tau_1 u_1||s_1| - \hat{k}_{v1}|s_1|] +\\ & (\hat{k}_{v1} - k_{v1})(-\kappa_1\hat{k}_{v1} + |s_1|) \le\\ & k_{d1}|s_1| + k_{\tau_1}|s_1| - k_{v1}|s_1| -\\ & \kappa_1\hat{k}_{v1}^2 + \kappa_1 k_{v1}\hat{k}_{v1} \le\\ & k_{d1}|s_1| + k_{\tau_1}|s_1| - k_{v1}|s_1| -\\ & \kappa_1(\hat{k}_{v1} - \frac{k_{v1}}{2})^2 + \frac{\kappa_1 k_{v1}^2}{4} \le\\ & -\underbrace{(k_{v1} - k_{d1} - k_{\tau_1})}_{\eta_1}|s_1| + \frac{\kappa_1 k_{v1}^2}{4}.\end{aligned} \tag{45}$$

It is quite clear that $\eta_1 > 0$. Clearly $\dot{V}_1 < 0$ if $|s_1| > \frac{\kappa_1 k_{v1}^2}{4\eta_{1_{\min}}}$. The decrease of V_1 eventually drives the trajectories of the closed loop system into $|s_1| < \frac{\kappa_1 k_{v1}^2}{4\eta_{1_{\min}}}$. Therefore the trajectories of the closed loop system are bounded[26, 33, 44, 48, 49]. □

Theorem 4. The convergence and boundness of the overall system can be obtained if we consider the tuning law (44), and virtual control and actual control law as given in (10), (15) and (20).

Proof. The overall Lyapunov function can be considered as a sum of all Lyapunov functions, i.e.

$$V = \sum_{i=1}^{n}\frac{1}{2}[s_i^2 + \frac{1}{\rho_i}\tilde{k}_{vi}^2]. \tag{46}$$

Taking the derivative of (46)

$$\dot{V} = \sum_{i=1}^{n}[s_i\dot{s}_i + \rho_i^{-1}\tilde{k}_{vi}\dot{\hat{k}}_{vi}]. \tag{47}$$

As proved in (45) and using the virtual and actual control law (10), (15) and (20), one can easily obtain

$$\dot{V} \le -\sum_{i=1}^{n}[\underbrace{(k_{vi} - k_{di} - k_{\tau_i})}_{\eta_i}|s_i| - \frac{\kappa_i k_{vi}^2}{4}]. \tag{48}$$

It is quite clear that $\upsilon_i > 0$. Clearly $\dot{V} < 0$ if $|s_i| > \frac{\kappa_i k_{vi}^2}{4\eta_{i_{\min}}}$. Therefore the trajectories of the closed loop system are bounded. In a similar way, since s_i is converging to zero then the convergence of σ_i is also assured by [33]. □

4 Simulation

The proposed adaptive step by step FOSM controller is simulated in MATLAB-Simulink by using ODE 4 solver with a fixed step size of 0.005 s.

Example 1. As an example, we consider the perturbed third order system[24]. The system equations are

$$\begin{aligned}\dot{x}_1 &= 2\sin(x_1) + 1.5x_2 + d_1(x_1, t)\\ \dot{x}_2 &= 0.8x_1x_2 + x_3 + d_2(\bar{x}_2, t)\\ \dot{x}_3 &= -1.5x_3^2 + 2u + d_3(x, t)\end{aligned} \tag{49}$$

where $d_1(x_1, t), d_2(\bar{x}_2, t), d_3(x, t)$ are given as

$$\begin{aligned}d_1(x_1, t) &= 0.2\sin(t) + 0.1x_1 + 0.12\\ d_2(\bar{x}_2, t) &= 0.3\sin(2t) + 0.2x_1 + 0.2x_2 - 0.4\\ d_3(x, t) &= 0.2\sin(2t) + 0.2x_1 + 0.3x_2 + 0.2x_3 + 0.3\end{aligned} \tag{50}$$

where $d_1(x_1, t)$, $d_2(\bar{x}_2, t)$ are the mismatched uncertainties and $d_3(x, t)$ is the matched uncertainty. The control objective is to track $x_d = 2\sin(0.15t) + 4\cos(0.1t) - 4$ by the output x_1. The initial conditions are taken as $x_0 = [3\ \ -2\ \ 4]^{\mathrm{T}}$.

Step 1. Since the order of the system with respect to the input is 3. Thus for the 1st step a 3rd order FOSM controller will be designed. The first full order sliding surface (6) is chosen by satisfying the conditions (7) and (8) as

$$\begin{aligned}s_1 = {} & \dot{\sigma}_1 + 11|\sigma_1|^{\frac{1}{2}}\mathrm{sgn}(\sigma_1) + 11|\dot{\sigma}_1|^{\frac{3}{5}}\mathrm{sgn}(\dot{\sigma}_1) + \\ & 2|\ddot{\sigma}_1|^{\frac{3}{4}}\mathrm{sgn}(\ddot{\sigma}_1)\end{aligned} \tag{51}$$

where $\sigma_1 = x_1 - x_d$. Hence the virtual control can be obtained as

$$\phi_1 = \frac{1}{1.5}[u_{eq1} + u_1]$$

where

$$\begin{aligned}u_{eq1} = {} & -2\sin(x_1) - 11|\sigma_1|^{\frac{1}{2}}\mathrm{sgn}(\sigma_1)\\ & -11|\dot{\sigma}_1|^{\frac{3}{5}}\mathrm{sgn}(\dot{\sigma}_1)\\ & -2|\ddot{\sigma}_1|^{\frac{3}{4}}\mathrm{sgn}(\ddot{\sigma}_1) + \dot{x}_d\end{aligned} \tag{52}$$

and u_1 can be calculated as $\dot{u}_1 + 0.05u_1 = v_1$ where $v_1 = -\hat{k}_{v1}\mathrm{sgn}(s_1)$. The value of $\hat{k}_{v1}$ can be estimated adaptively if the following tuning function is chosen $\dot{\hat{k}}_{v1} = -\hat{k}_{v1} + 5.5|s_1|$.

Step 2. Now the relative position of 2nd state equation with respect to the control input is 2. Hence a second order FOSM controller will be designed. Now the second full order sliding surface satisfying the conditions (7) and (8) is chosen as

$$s_2 = \dot{\sigma}_2 + 10|\sigma_2|^{\frac{1}{2}}\mathrm{sgn}(\sigma_2) + 2|\dot{\sigma}_2|^{\frac{2}{3}}\mathrm{sgn}(\dot{\sigma}_2) \tag{53}$$

where $\sigma_2 = x_2 - \phi_1$, and the second virtual control law can be obtained as

$$\begin{aligned}\phi_2 &= [u_{eq2} + u_2]\\ u_{eq2} &= -0.8x_1x_2 - 10|\sigma_2|^{\frac{1}{2}}\mathrm{sgn}(\sigma_2) - \\ & \quad 2|\dot{\sigma}_2|^{\frac{2}{3}}\mathrm{sgn}(\dot{\sigma}_2) + \dot{\phi}_1\end{aligned} \tag{54}$$

where u_2 can be calculated as $\dot{u}_2 + 0.05u_2 = v_2$ where $v_2 = -\hat{k}_{v2}\mathrm{sgn}(s_2)$. The value of $\hat{k}_{v2}$ can be estimated adaptively if the following tuning function is chosen $\dot{\hat{k}}_{v2} = -\hat{k}_{v2} + 6.1|s_2|$.

Step 3. Now the last state equation of system (49) has a position 1, so a first order FOSM controller will be designed. Last full order sliding surface can be designed as

$$s_3 = \dot{\sigma}_3 + 0.5|\sigma_3|^{\frac{1}{2}}\mathrm{sgn}(\sigma_3) \tag{55}$$

where σ_3 is defined as $\sigma_3 = x_3 - \phi_2$, and the actual control law can be obtained as

$$\begin{aligned}u &= \frac{1}{2}[u_{eq3} + u_3]\\ u_{eq3} &= 1.5x_3^2 - 0.5|\sigma_3|^{\frac{1}{2}}\mathrm{sgn}(\sigma_3) + \dot{\phi}_2\end{aligned} \tag{56}$$

and $\dot{u}_3 + 0.05u_3 = v_3$ where $v_3 = -\hat{k}_{v3}\mathrm{sgn}(s_3)$ and the estimation of $\hat{k}_{v3}$ is given by $\dot{\hat{k}}_{v3} = -0.5\hat{k}_{v3} + 16|s_3|$.

The output tracking response and the system states by using the step by step adaptive FOSM controller are shown in Fig. 2, which shows that the proposed controller guaranteed accurate tracking in spite of the mismatched uncertainties.

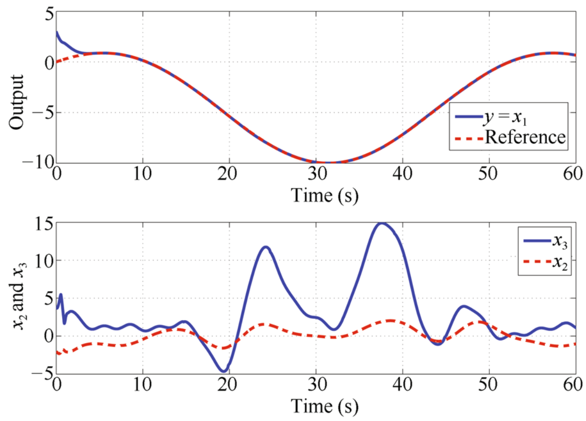

Fig. 2 State trajectory using the proposed controller

Fig. 3 depicts the input signal using proposed control law. It is clear that the undesired chattering in the control input is reduced effectively as compared to the method proposed in [24].

The estimated parameters $\hat{k}_{v1}$, $\hat{k}_{v2}$ and $\hat{k}_{v3}$ with the initial conditions $\hat{k}_{v1}(0) = 1$, $\hat{k}_{v2}(0) = 1$, and $\hat{k}_{v3}(0) = 1$ are shown in Fig. 4.

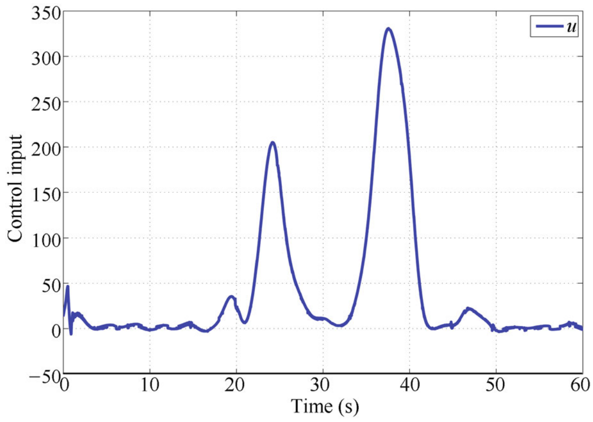

Fig. 3 Control signal

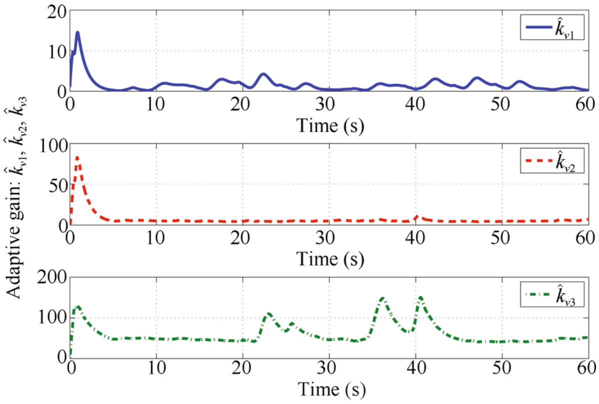

Fig. 4 Adaptive gain

It is clear that the parameters are bounded without using the boundary layer technique (43).

Example 2. Now to observe the chattering reduction effect of the proposed controller, the following example of inverted pendulum[50] with mismatched uncertainty is considered. Control objective is to track the reference signal chosen as

$$\begin{aligned}
&x_d = \sin(t)\\
&\dot{x}_1 = x_2 + d_1\\
&\dot{x}_2 = \frac{g\sin x_1 - (\dfrac{mlx_2^2\cos x_1 \sin x_1}{m_c} + m)}{l[\dfrac{4}{3} - (\dfrac{m\cos^2 x_1}{m_c} + m)]} + \\
&\qquad \frac{\dfrac{\cos x_1}{m_c} + m}{l[\dfrac{4}{3} - (\dfrac{m\cos^2 x_1}{m_c} + m)]} u + d_2\\
&y = x_1
\end{aligned} \tag{57}$$

where d_1 and d_2 are given as

$$\begin{aligned}
&d_1 = 0.1\cos(4t)\\
&d_2 = 0.4\sin(4t) + \sin(10x_1) + \cos(x_2).
\end{aligned} \tag{58}$$

The parameters of the single inverted pendulum are tabulated in Table 1.

Table 1 Parameters of the single inverted pendulum

Variable name	Description	Values
g	Gravitational constant	9.8 ms^{-2}
m_c	Mass of the cart	1 kg
m	Mass of the pendulum	0.1 kg
l	Effective length of the pendulum	0.5 m
x_1	Swing angle	state
x_2	Swing speed	state

The proposed step by step adaptive FOSM controller is now compared with the controller desgned by Estrada and Fridman[23] where $\phi_1 = u_1$, $\dot{u}_1 = -3\frac{\dot{\sigma}_1 - |\sigma_1|^{\frac{1}{2}}}{|\dot{\sigma}_1| + |\sigma_1|^{\frac{1}{2}}}$, $\sigma_1 = x_1 - x_d$, $\sigma_2 = x_2 - \phi_1$ and the final control law is obtained by

$$\begin{aligned}
u = &\left[\frac{\dfrac{\cos x_1}{m_c} + m}{l[\dfrac{4}{3} - (\dfrac{m\cos^2 x_1}{m_c} + m)]}\right]^{-1} \times\\
&\left[-\frac{g\sin x_1 - (\dfrac{mlx_2^2\cos x_1 sin x_1}{m_c} + m)}{l[\dfrac{4}{3} - (\dfrac{m\cos^2 x_1}{m_c} + m)]} - \right.\\
&\left. 5\mathrm{sgn}(\sigma_2)\right].
\end{aligned} \tag{59}$$

The output signal and the control input are shown in Figs. 5 and 6, which show that the controller given by (59) achieves good tracking performance with rapid convergence but the major drawback is the high frequency chattering present in the control input. Moreover, another design constraint is that the uncertainty are needed to be known a priori.

Now for comparison a step by step adaptive FOSM controller is designed for inverted pendulum (57). Since the inverted pendulum (57) has a relative position 2, so for the first step a second order FOSM controller will be designed. The sliding surface is chosen as

$$s_1 = \dot{\sigma}_1 + 5|\sigma_1|^{\frac{1}{2}}\mathrm{sgn}(\sigma_1) + 1.5|\dot{\sigma}_1|^{\frac{2}{3}}\mathrm{sgn}(\dot{\sigma}_1)$$

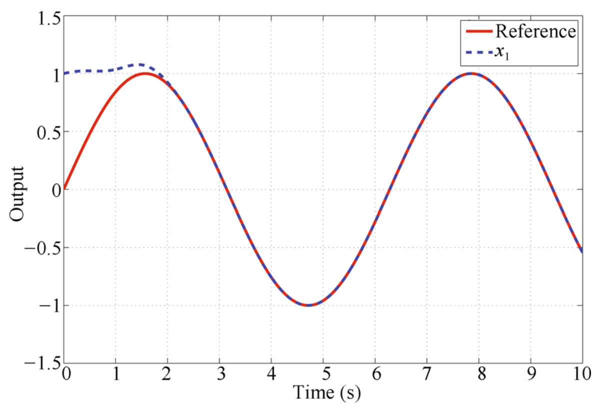

Fig. 5 Reference trajectory versus actual trajectory using the method in [23]

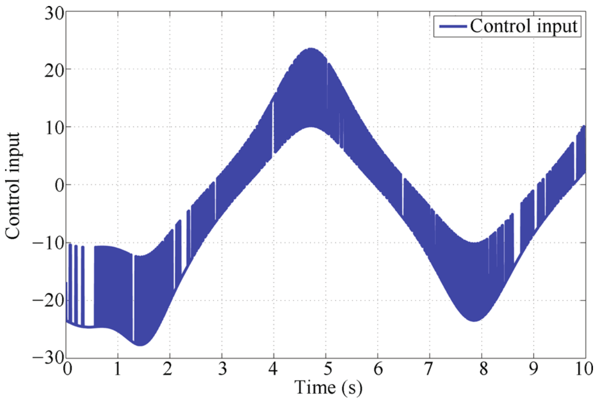

Fig. 6 Control signal using the method in [23]

where $\sigma_1 = x_1 - x_d$, and the first virtual control law is obtained as

$$\phi_1 = [u_{eq1} + u_1]$$
$$u_{eq1} = -5|\sigma_1|^{\frac{1}{2}}\mathrm{sgn}(\sigma_1) - 1.5|\dot{\sigma}_1|^{\frac{2}{3}}\mathrm{sgn}(\dot{\sigma}_1) + \dot{x}_d$$

where u_1 can be calculated as $\dot{u}_1 + 0.1u_1 = v_1$ where $v_1 = -\hat{k}_{v1}\mathrm{sgn}(s_1)$. The value of $\hat{k}_{v1}$ can be estimated with the tuning function chosen as $\dot{\hat{k}}_{v1} = -\hat{k}_{v1} + 6.1|s_1|$, where $\hat{k}_{v1}(0) = 1$.

Now s_2 is considered as

$$s_2 = \dot{\sigma}_2 + 0.5|\sigma_2|^{\frac{1}{2}}\mathrm{sgn}(\sigma_2)$$

where σ_2 is defined as $\sigma_2 = x_2 - \phi_1$, and the actual control law can be obtained as

$$u = \left[\frac{\dfrac{\cos x_1}{m_c} + m}{l[\dfrac{4}{3} - (\dfrac{m\cos^2 x_1}{m_c} + m)]}\right]^{-1}[u_{eq2} + u_2]$$

where

$$u_{eq2} = -\frac{g\sin x_1 - (\dfrac{mlx_2^2\cos x_1 \sin x_1}{m_c} + m)}{l[\dfrac{4}{3} - (\dfrac{m\cos^2 x_1}{m_c} + m)]} - 0.5|\sigma_2|^{\frac{1}{2}}\mathrm{sgn}(\sigma_2) + \dot{\phi}_1.$$

Also $\dot{u}_2 + 0.1u_2 = v_2$, where $v_2 = -\hat{k}_{v2}\mathrm{sgn}(s_2)$ and the estimation of $\hat{k}_{v2}$ is given by $\dot{\hat{k}}_{v2} = -0.5\hat{k}_{v2} + 4|s_2|$, where $\hat{k}_{v2}(0) = 1$.

The output signal and the control input by using the proposed adaptive FOSM controller are shown in Figs. 7 and 8, which show that the tracking is accurate and the control signal is smooth as compared to Fig. 6[23]. The major benefits are that the proposed controller does not required any prior knowledge about the uncertainties and control signal is smooth which is more suitable for electromechanical systems. The convergence of estimated parameter is shown in Fig. 9.

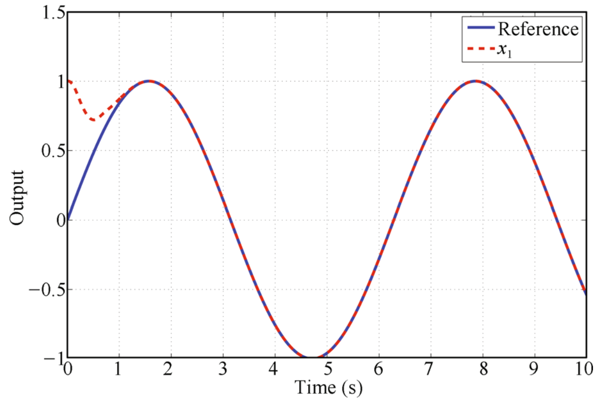

Fig. 7 Reference trajectory versus actual trajectory using the proposed method

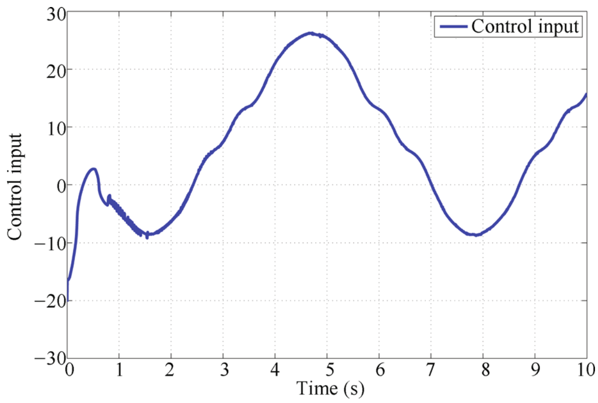

Fig. 8 Control signal using the proposed method

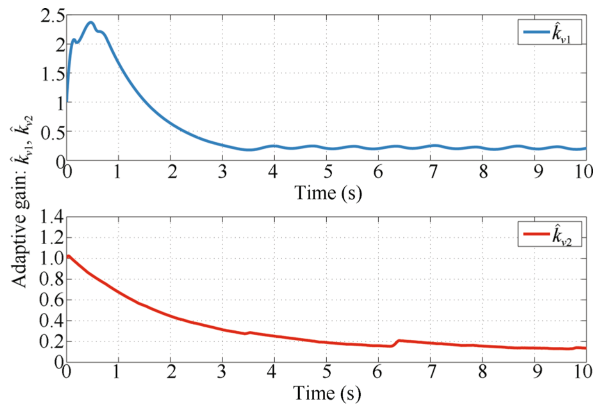

Fig. 9 Adaptive gains: $\hat{k}_{v1}$ and $\hat{k}_{v2}$

5 Conclusions

A step by step adaptive full order sliding mode controller (FOSM) controller is presented for nonlinear strict feedback systems with mismatched uncertainties. The proposed algorithm provides a finite time tracking of smooth signal. At each step a full order sliding surface is designed to produce the finite time convergence. To deal with the unknown bounded uncertainties an adaptive tuning rule is designed. However, the knowledge about the upper bound of the system uncertainties is not required to be known a priori.

Simulation results demonstrate that the proposed control strategy is more efficient and successful in reducing the undesired chattering in the control input while ensuring a satisfactory tracking performance in presence of mismatched uncertainty.

References

[1] W. B. Gao, J. C. Hung. Variable structure control of nonlinear systems: A new approach. *IEEE Transactions on Industrial Electronics*, vol. 40, no. 1, pp. 45–50, 1993.

[2] C. Edwards, S. K. Spurgeon. *Sliding Mode Control: Theory and Applications*, London, UK: Taylor & Francis, 1998.

[3] M. Ö. Efe, C. Ünsal, O. Kaynak, X. H. Yu. Variable structure control of a class of uncertain systems. *Automatica*, vol. 40, no. 1, pp. 59–64, 2004.

[4] H. Romdhane, K. Dehri, A. S. Nouri. Second order sliding mode control for discrete decouplable multivariable systems via input-output models. *International Journal of Automation and Computing*, vol. 12, no. 6, pp. 630–638, 2015.

[5] S. Mahjoub, F. Mnif, N. Derbel. Second-order sliding mode approaches for the control of a class of underactuated systems. *International Journal of Automation and Computing*, vol. 12, no. 2, pp. 134–141, 2015.

[6] V. I. Utkin, J. Guldner, J. Shi. *Sliding Mode Control in Electromechanical Systems*, London, UK: Taylor & Francis, 1999.

[7] S. K. Spurgeon, R. Davies. A nonlinear control strategy for robust sliding mode performance in the presence of unmatched uncertainty. *International Journal of Control*, vol. 57, no. 5, pp. 1107–1123, 1993.

[8] W. J. Cao, J. X. Xu. Nonlinear integral-type sliding surface for both matched and unmatched uncertain systems. *IEEE Transactions on Automatic Control*, vol. 49, no. 8, pp. 1355–1360, 2004.

[9] H. H. Choi. An explicit formula of linear sliding surfaces for a class of uncertain dynamic systems with mismatched uncertainties. *Automatica*, vol. 34, no. 8, pp. 1015–1020, 1998.

[10] T. R. Oliveira, A. J. Peixoto, L. Hsu. Peaking free output-feedback exact tracking of uncertain nonlinear systems via dwell-time and norm observers. *International Journal of Robust and Nonlinear Control*, vol. 23, no. 5, pp. 483–513, 2013.

[11] C. W. Tao, M. L. Chan, T. T. Lee. Adaptive fuzzy sliding mode controller for linear systems with mismatched time-varying uncertainties. *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)*, vol. 33, no. 2, pp. 283–294, 2003.

[12] H. H. Choi. LMI-based sliding surface design for integral sliding mode control of mismatched uncertain systems. *IEEE Transactions on Automatic Control*, vol. 52, no. 4, pp. 736–742, 2007.

[13] Y. Chang, C. C. Cheng. Design of adaptive sliding surfaces for systems with mismatched perturbations to achieve asymptotical stability. *IET Control Theory & Application*, vol. 1, no. 1, pp. 417–421, 2007.

[14] C. C. Wen, C. C. Cheng. Design of sliding surface for mismatched uncertain systems to achieve asymptotical stability. *Journal of the Franklin Institute*, vol. 345, no. 8, pp. 926–941, 2008.

[15] C. C. Cheng, Y. T. Chang. Design of decentralised adaptive sliding mode controllers for large-scale systems with mismatched perturbations. *International Journal of Control*, vol. 81, no. 10, pp. 1507–1518, 2008.

[16] J. Yang, S. H. Li, J. Y. Su, X. H. Yu. Continuous nonsingular terminal sliding mode control for systems with mismatched disturbances. *Automatica*, vol. 49, no. 7, pp. 2287–2291, 2013.

[17] D. Ginoya, P. D. Shendge, S. B. Phadke. State and extended disturbance observer for sliding mode control of mismatched uncertain systems. *Journal of Dynamic Systems, Measurement, and Control*, vol. 137, no. 7, pp. 074502, 2015.

[18] D. Ginoya, P. D. Shendge, S. B. Phadke. Sliding mode control for mismatched uncertain systems using an extended disturbance observer. *IEEE Transactions on Industrial Electronics*, vol. 61, no. 4, pp. 1983–1992, 2014.

[19] A. Ferrara, L. Giacomini, C. Vecchio. Control of nonholonomic systems with uncertainties via second-order sliding modes. *International Journal of Robust and Nonlinear Control*, vol. 18, no. 4–5, pp. 515–528, 2008.

[20] G. Bartolini, A. Ferrara, L. Giacomini, E. Usai. A combined backstepping/second order sliding mode approach to control a class of nonlinear systems. In *Proceedings of IEEE International Workshop on Variable Structure Systems*, IEEE, Tokyo, Japan, pp. 205–210, 1996.

[21] A. Ferrara, L. Giacomini. On multi-input backstepping design with second order sliding modes for a class of uncertain nonlinear systems. *International Journal of Control*, vol. 71, no. 5, pp. 767–788, 1998.

[22] F. Ikhouane, M. Krstic. Robustness of the tuning functions adaptive backstepping design for linear systems. *IEEE Transactions on Automatic Control*, vol. 43, no. 3, pp. 431–436, 1998.

[23] A. Estrada, L. Fridman. Quasi-continuous HOSM control for systems with unmatched perturbations. *Automatica*, vol. 46, no. 11, pp. 1916–1919, 2010.

[24] A. Estrada, L. M. Fridman. Integral HOSM semiglobal controller for finite-time exact compensation of unmatched perturbations. *IEEE Transactions on Automatic Control*, vol. 55, no. 11, pp. 2645–2649, 2010.

[25] A. Estrada, A. Loria, R. Santiesteban, L. Fridman. Cascaded-based stabilization of time-varying systems using second-order sliding modes. *IMA Journal of Mathematical Control and Information*, vol. 30, no. 1, pp. 115–128, 2013.

[26] S. Mondal, C. Mahanta. Chattering free adaptive multivariable sliding mode controller for systems with matched and mismatched uncertainty. *ISA Transactions*, vol. 52, no. 3, pp. 335–341, 2013.

[27] Y. Feng, F. L. Han, X. H. Yu. Chattering free full-order sliding-mode control. *Automatica*, vol. 50, no. 4, pp. 1310–1314, 2014.

[28] L. Y. Fang, T. S. Li, Z. F. Li, R. H. Li. Adaptive terminal sliding mode control for anti-synchronization of uncertain chaotic systems. *Nonlinear Dynamics*, vol. 74, no. 4, pp. 991–1002, 2013.

[29] J. T. Fei, M. Y. Xin. Adaptive fuzzy sliding mode control of MEMS gyroscope sensor using fuzzy switching approach. *Journal of Dynamic Systems, Measurement, and Control*, vol. 137, no. 5, pp. 051002, 2015.

[30] S. Mondal, C. Mahanta. Adaptive second-order sliding mode controller for a twin rotor multi-input-multi-output system. *IET Control Theory & Applications*, vol. 6, no. 14, pp. 2157–2167, 2012.

[31] S. Mondal, C. Mahanta. Adaptive second order terminal sliding mode controller for robotic manipulators. *Journal of the Franklin Institute*, vol. 351, no. 4, pp. 2356–2377, 2014.

[32] J. T. Fei, H. F. Ding. Adaptive sliding mode control of dynamic system using RBF neural network. *Nonlinear Dynamics*, vol. 70, no. 2, pp. 1563–1573, 2012.

[33] Y. Q. Xia, Z. Zhu, M. Y. Fu, S. Wang. Attitude tracking of rigid spacecraft with bounded disturbances. *IEEE Transactions on Industrial Electronics*, vol. 58, no. 2, pp. 647–659, 2011.

[34] M. B. R. Neila, D. Tarak. Adaptive terminal sliding mode control for rigid robotic manipulators. *International Journal of Automation and Computing*, vol. 8, no. 2, pp. 215–220, 2011.

[35] Y. J. Huang, T. C. Kuo, S. H. Chang. Adaptive sliding-mode control for nonlinear systems with uncertain parameters. *IEEE Transactions on Systems, Man, and Cybernatics, Part B (Cybernetics)*, vol. 38, no. 2, pp. 534–539, 2008.

[36] P. Li, Z. Q. Zheng. Robust adaptive second-order sliding-mode control with fast transient performance. *IET Control Theory & Applications*, vol. 6, no. 2, pp. 305–312, 2012.

[37] S. Mobayen. An adaptive chattering-free PID sliding mode control based on dynamic sliding manifolds for a class of uncertain nonlinear systems. *Nonlinear Dynamics*, vol. 82, no. 1, pp. 53–60, 2015.

[38] V. I. Utkin, A. S. Poznyak. Adaptive sliding mode control with application to super-twist algorithm: Equivalent control method. *Automatica*, vol. 49, no. 1, pp. 39–47, 2013.

[39] C. Edwards, Y. B. Shtessel. Adaptive continuous higher order sliding mode control. *Automatica*, vol. 65, pp. 183–190, 2016.

[40] M. S. S. Ahmed, P. Zhang, Y. J. Wu. Position control of synchronous motor drive by modified adaptive two-phase sliding mode controller. *International Journal of Automation and Computing*, vol. 5, no. 4, pp. 406–412, 2008.

[41] A. Levant. Higher-order sliding modes, differentiation and output-feedback control. *International Journal of Control*, vol. 76, no. 9–10, pp. 924–941, 2003.

[42] S. P. Bhat, D. S. Bernstein. Geometric homogeneity with applications to finite-time stability. *Mathematics of Control, Signals and Systems*, vol. 17, no. 2, pp. 101–127, 2005.

[43] M. Defoort, T. Floquet, A. Kokosy, W. Perruquetti. A novel higher order sliding mode control scheme. *System & Control Letters*, vol. 58, no. 2, pp. 102–108, 2009.

[44] F. Plestan, Y. Shtessel, V. Brégeault, A. Poznyak. New methodologies for adaptive sliding mode control. *International Journal of Control*, vol. 83, no. 9, pp. 1907–1919, 2010.

[45] D. Zhao, S. Li, Q. Zhu, F. Gao. Robust finite-time control approach for robotic manipulators. *IET Control Theory & Applications*, vol. 4, no. 1, pp. 1–15, 2010.

[46] S. Mobayen. Finite-time tracking control of chained-form nonholonomic systems with external disturbances based on recursive terminal sliding mode method. *Nonlinear Dynamics*, vol. 80, no. 1–2, pp. 669–683, 2015.

[47] S. Ghabraei, H. Moradi, G. Vossoughi. Multivariable robust adaptive sliding mode control of an industrial boiler-turbine in the presence of modeling imprecisions and external disturbances: A comparison with type-I servo controller. *ISA Transactions*, vol. 58, pp. 398–408, 2015.

[48] M. Taleb, F. Plestan, B. Bououlid. An adaptive solution for robust control based on integral high-order sliding mode concept. *International Journal of Robust and Nonlinear Control*, vol. 25, no. 8, pp. 1201–1213, 2015.

[49] Y. M. Li, Q. S. Xu. Adaptive sliding mode control with perturbation estimation and PID sliding surface for motion tracking of a piezo-driven micromanipulator. *IEEE Transactions on Control System and Technology*, vol. 18, no. 4, pp. 798–810, 2010.

[50] Q. Zong, Z. S. Zhao, J. Zhang. Brief paper: Higher order sliding mode control with self-tuning law based on integral sliding mode. *IET Control Theory & Applications*, vol. 4, no. 7, pp. 1282–1289, 2010.

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