International Journal of Automation and Computing 1 (2004) 51-62

Adaptive sliding mode control for MIMO nonlinear systems based on a fuzzy logic scheme

Feng Qiao, Quanmin Zhu*, Allen FT Winfield, Chris Melhuish
Intelligent Autonomous Systems Laboratory, Engineering and Mathematical Sciences University of the West of England, Frenchay Campus, Coldharbour Lane, Bristol BS16 1QY, UK

Abstract: In this study an indirect adaptive sliding mode control (SMC) based on a fuzzy logic scheme is proposed to strengthen the tracking control performance of a general class of multi-input/multi-output (MIMO) nonlinear uncertain systems. Combining reaching law and fuzzy universal approximation, the proposed design procedure combines the advantages of fuzzy logic, adaptive and sliding mode control. The stability of the control systems is proved in the sense of a Lyapunov second stability theorem. Two simulation studies are presented to demonstrate the effectiveness of our new hybrid control algorithm.

Keywords: Fuzzy logic control (FLC), sliding mode control (SMC), multi-input/multi-output (MIMO), nonlinear uncertain systems, adaptive control, variable structure control (VSS).

1 Introduction

Most complex dynamic systems exhibit nonlinearities and uncertainties that are difficult to describe with mathematical models. In addition, the mathematical treatment of non-linear systems compared to linear systems is not well understood. Although control theory is well developed for linear systems, there is still no universal control solution for many kinds of nonlinear system.

Since the initiative work of Mandani[1] based on the fuzzy set theory of Zadeh[2], fuzzy logic control (FLC) has been widely applied to deal with many practical problems. FLC is easy to understand and simple and cheap to implement. FLC is used to utilise the qualitative knowledge of a system to design a practical controller. FLC is generally applicable to plants that are ill modelled, but have qualitative knowledge from experienced operators available to aid design. FLC is particularly suitable for systems with uncertain and/or complex dynamics. However, despite its great successes in commercial and industrial practices, FLC has been the target of criticism for its lack of systematic design, mathematical rigor, and concrete stability analyses.

Sliding mode control (SMC) for variable structure systems (VSS) is a well-applied technique for systems whose accurate mathematical models are difficult to obtain. It was first proposed and elaborated in the early 1950s in the former Soviet Union by Emelyanov and several co-researchers[3~5]. From then on, SMC has expanded into a general design method examined for a wide spectrum of system types including nonlinear, multi-input/multi-output (MIMO), discrete time, large scale and infinite dimension and stochastic systems. To resolve real problems SMC has been adopted in a wide variety of engineering systems. A great deal of efforts has been put on establishing both theoretical VSS concepts and practical applications. A comprehensive survey paper on VSS was published by Hung, *et al.*[6] which is highly quoted. Some of the concepts and theoretical advances in VSS are covered in some literatures, such as DeClarlo, *et al.*[7], Slotine and Li[8], Utkin[9], and Zinober[10]. Professor Vadim Utkin delivered a paper on the 6$^{\text{th}}$ IEEE Workshop on Variable Structure Systems entitled "VSS Premise in XX Century: Evidence of a Witness"[11], which sought to trace the evolution of VSS theory from its early developments in the former USSR to its current state, and highlighted the emergence and later abandonment of certain trend/paradigms as the theory evolved.

The most distinguishing property of SMC is its robustness, that is, closed loop systems are completely insensitive to modelling uncertainties, time varying parameter fluctuations, and external disturbances. Strong robustness comes from the fact that SMC makes use of a designed sliding surface in state space, and produces switched control settings with consideration for observed system input-output behaviour, a boundary of modelling uncertainties, and unknown disturbances[8]. In spite of its wide applications, pure SMC has exposed some obvious disadvantages. The first is chatter, which is highly undesirable in practical implementations, because it may excite high frequency dynamics in unmodeled parts of a system and cause system instabilities or even disasters. The other

Manuscript received November 1, 2002; revised July 15, 2003.
*Corresponding author. *E-mail address*: Quan.Zhu@UWE.ac.uk

is that, in a practical implementation, it is difficult to identify an approximation of system models as well as model uncertain boundaries and external disturbances. One commonly used chatter attenuating method in conventional control is to introduce a boundary layer in the vicinity of a sliding surface which achieves a trade-off between tracking precision and robustness in a system[8]. Another method is to integrate FLC into SMC[12]. FLC can perform a similar function to that of a boundary layer.

Recently the synthesis of algorithms of modern control theory and artificial intelligence has been studied to upgrade the performance of conventional SMC. Kaynak *et al.*[13] published a survey paper on the fusion of computationally intelligent methodologies and SMC. Fuzzy sliding mode control, which takes the features of both SMC and FLC to overcome the disadvantages of chatter and enhance the robustness of controllers, is one such example[14~16].

Meanwhile adaptive control techniques have also successfully advanced in tackling control problems for uncertain nonlinear systems[17,18]. Many contributions have been made in the integration of FLC and SMC based on adaptive schemes, most of which take the benefits of fuzzy universal approximation theorem[19] to incorporate expert information systematically and guarantee various stable criteria. In general the parameters and structures of plant to be controlled present large uncertainties and variations. The objective of the design of a controller is to maintain persistent performance of system in the presence of these uncertainties and variations, with adaptive control aimed at achieving this objective. Therefore advanced fuzzy control also should be adaptive in order to maintain desired performance. Fuzzy adaptive control, which takes linguistic information from human operators as an advantage over conventional adaptive control, is especially suitable for systems which are subject to high degrees of uncertainty.

The research work of Su and Stepanenko[20], Yoo and Ham[21], Tong *et al.*[22], Chai and Tong[23], Wang *et al.*[24], and Chan *et al.*[25] apply fuzzy basis functions to approximate unknown system parameters in the design of SMC control law, in which the weights of fuzzy basis functions are adaptively adjusted according to fuzzy universal approximation theorem, and proof of system stability is presented accordingly. However these schemes have been applied only to a class of single-input/single-output nonlinear systems. Their principles should also be extendable to MIMO systems.

Chang[26] proposed a hybrid adaptive robust tracking control scheme for MIMO nonlinear systems based on a combination of H^∞, VSS control algorithm and FLC design where system uncertainties are approximated with adaptive fuzzy approximators. Li and Tong[27] and Tong and Li[28] published their studies to deal with MIMO plants with unavailable state variables based on fuzzy adaptive SMC.

In this study, based on fuzzy universal approximation theorem an indirect fuzzy adaptive control scheme is proposed for integration with SMC for the control of MIMO non-linear systems which have modelling uncertainties and parameter fluctuations. The reaching law method proposed by Hung and Gao[6] is adopted here. The stability of the control algorithm is proved in terms of a Lyapunov theorem, with tracking error converging to the vicinity of zero.

The remainder of the paper is organised as follows. Section 2 provides the objective of system control. A general form of MIMO nonlinear system is presented in a set of differential equations; the preliminary knowledge of sliding mode control and fuzzy systems together with fuzzy basis functions and fuzzy universal approximation theorem and some related definitions and lemmas are briefly reviewed. Section 3 proposes the design scheme for a MIMO fuzzy adaptive sliding mode controller for nonlinear systems. The stability of the scheme is proved in terms of a Lyapunov theorem in two cases. Section 4 provides two case studies to demonstrate the effectiveness of the proposed control scheme in simulation. Section 5 draws some conclusions.

2 Preliminaries

2.1 Problem statement

Consider a general form of MIMO nonlinear system as follows[27]:

$$\begin{aligned} &\dot{x}_i^{(n_i)} = f_i(x) + \sum_{j=1}^{p} g_{ij}(x)u_j + d_i(t) \\ &y_i = x_i \quad (i = 1\cdots p) \end{aligned} \tag{1}$$

where $x_1, \cdots, x_1^{(n_1-1)}, \cdots, x_p, \cdots, x_p^{(n_p-1)}$ are system states, and $y_1, \cdots, y_p$ are outputs of the system, $u_1, \cdots, u_p$ are inputs to the system, $f_1(x), \cdots, f_p(x)$, are system functions, $g_{ij}(x)$ are system gains and $d_1, \cdots, d_p$ are system disturbances.

The model (1) can be rewritten in matrix form as:

$$\begin{aligned} &\dot{x} = Ax + B[F(x) + G(x)u + d] \\ &y = Cx \end{aligned} \tag{2}$$

where:

$$\begin{aligned} &x = [x_1, \cdots, x_1^{(n_1-1)}, \cdots, x_p, \cdots, x_p^{(n_p-1)}] \\ &y = [y_1, \cdots, y_p]^T \\ &A = diay[A_1, \cdots, A_p] \\ &B = diay[B_1, \cdots, B_p] \end{aligned}$$

$$C = diay[C_1, \cdots, C_p]$$
$$u = [u_1, \cdots, u_p]^T$$
$$d = [d_1, \cdots, d_p]^T$$

and:

$$A_i = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{n_i \times n_i} \quad (i = 1, \cdots, p)$$
$$B_i = [0 \quad 0 \quad \cdots \quad 0 \quad 1]^T_{n_i}$$
$$C_i = [1 \quad 0 \quad \cdots \quad 0 \quad 0]_{n_i}$$
$$F(x) = [f_1(x), \quad \cdots, \quad f_p(x)]^T$$
$$G(x) = [G_1, \quad \cdots, \quad G_p]^T$$
$$G_i(x) = [g_{i1}, \quad \cdots, \quad g_{ip}]^T \quad (i = 1, \cdots, p)$$

The following assumptions are made in the rest of the paper:

A1. The system state vector $x \in R^n (n = \sum_{i=1}^{p} n_i)$ in (2) is measurable.

A2. The system function vector $F(x) = [f_1(x), \cdots, f_p(x)]^T$ is not known exactly but is bounded, that is $\| F(x) \| \leqslant F_0(x)$, where $F_0(x)$ is a known vector, and its elements are smooth, that is, $f_i(x)$s$(i = 1 \cdots n)$ in $F(x)$ are smooth functions.

A3. The system gain matrix $G(x)$ is not known exactly, but bounded by $0 < G_m(x) \leqslant G(x) \leqslant G_M(x)$, where $G_m(x)$ and $G_M(x)$ are known matrices, $G(x)$ is nonsingular, and $g_{ij}(x)$s$(i, j = 1 \cdots p, p)$ in $G(x)$ are smooth functions.

The objective of this research is to design an adaptive controller for the general form of non-linear MIMO system of (2) despite plant uncertainties and external disturbances, which guarantees boundedness of all the variables of a closed loop system and outputs tracking of the given reference signal $y_r = \lfloor y_{r_1}, \cdots, y_{r_p} \rfloor^T$. Let $Y_r = [y_{r_1}, \cdots, y_{r_1}^{(n_1-1)}, \cdots, y_{r_p}, \cdots, y_{r_p}^{(n_p-1)}]^T$ and assume that there is a compact set Ω_r such that $Y_r(t) \in \Omega_r, \forall t \geqslant 0$.

2.2 Sliding mode control

The design of a sliding mode control system can be divided into two steps: step one is the selection of a proper sliding function while step two is the design of a control law. In step one, a sliding (or switching) function vector with p dimensions will be selected for a system with p inputs. In step two, a proper sliding mode control law vector will be designed to meet the requirements of a sliding mode reaching condition.

The sliding function vector is designed as:

$$S = [s_1, \cdots, s_p]^T \tag{3}$$

where $s_i = C_i^T E_i (i = 1, \cdots, p)$, and $C_i = [c_{i1}, \cdots, c_{i(n_i-1)}, 1]^T$ is the Hurwitzian coefficient vector, $E_i = [e_i, \cdots, e_i^{(n_i-1)}]^T$ is a tracking error vector whose elements are defined by the following equations:

$$\begin{aligned} e_i &= x_i - y_{r_i} \\ &\vdots \\ e_i^{(n_i-1)} &= x_i^{(n_i-1)} - y_{r_i}^{(n_i-1)} \end{aligned} \quad (i = 1, \cdots, p) \tag{4}$$

The following equations can be obtained with a derivative of (3):

$$\dot{S} = [\dot{s}_1, \cdots, \dot{s}_p]^T \tag{5}$$

and:

$$\begin{aligned} \dot{s}_i = C_i \dot{E}_i &= \sum_{j=1}^{n_i} c_{i,j} e_i^{(j)} \\ &= \sum_{j=1}^{n_i-1} (c_{i,j} e_i^{(j)}) + e_i^{(n_i)} \\ &= \sum_{j=1}^{n_i-1} (c_{i,j} e_i^{(j)}) + f_i(x) + \sum_{j=1}^{p} g_{ij}(X) u_j + d_i(t) - y_{r_i}^{(n_i)} \\ &= R_i + f_i(x) + \sum_{j=1}^{p} g_{ij}(X) u_j + d_i(t) - y_{r_i}^{(n_i)} \end{aligned} \tag{6}$$

where $R_i = \sum_{j=1}^{n_i-1} (c_{i,j} e_i^{(j)})$.

Equation (5) can then be written in matrix form as:

$$\dot{S} = R + F(x) + G(x)u + d - y' \tag{7}$$

where $R = [R_1, \cdots, R_p]^T$ and $y' = [y_{r_1}^{(n_1)}, \cdots, y_{r_p}^{(n_p)}]^T$.

Using the reaching law method[6]:

$$\dot{S} = -H(S)$$

and equation (7), the following can be obtained:

$$R + F(x) + G(x)u + d - y' = -H(S)$$

where:

$$H(S) = Q\text{sgn}(S) + KW_H(S)$$

and:

$$Q = diag\lfloor q_1, \cdots, q_p \rfloor, \quad q_i > 0, (i = 1, \cdots, p)$$
$$\text{sgn}(S) = [\text{sgn}(s_1), \cdots, \text{sgn}(s_p)]^T$$
$$K = diag\lfloor k_1, \cdots, k_p \rfloor, \quad k_i > 0, (i = 1, \cdots, p)$$
$$W_H(S) = [w_{h1}(s_1), \cdots, w_{hp}(s_p)]^T$$

in which q_i, $k_i(i = 1, \cdots, p)$ are designed parameters and $s_i w_{hi}(s_i) > 0$ and $w_{hi}(0) = 0$.

Here, the reaching law with a constant plus proportional rate is applied, that is $w_{hi}(s_i) = s_i(i = 1, \cdots, p)$.

If the system function $F(x)$ and the control gain matrix $G(x)$ are known, and $G(x)$ is nonsingular, that is $G(x)$ is invertible, and the external disturbance x is known beforehand, from (8), the control law vector u can easily be determined by:

$$u = -G^{-1}(x)(R + F(x) + d - y' + H(S)) \tag{9}$$

It is easy to prove that the control law so designed can force system state to track the given reference, and that the tracking error will converge to the vicinity of zero within a finite time period.

But in practical applications, it is usually difficult to exactly model a plant in mathematical equations, or sometimes, it is impossible to obtain a model. In order to design the control law in (9), the fuzzy universal approximation theorem is applied to approximate its parameters.

2.3 Fuzzy system

A fuzzy system consists of four principle parts as shown in Fig.1, which are a fuzzifier, a fuzzy rule base, a fuzzy inference engine and a defuzzifier. The four parts of a fuzzy system will be detailed here with a multi-input/single-output structure: $U \subset R^n \rightarrow R$, where U is a compact set. A multi-output system can be separated into a group of single-output systems.

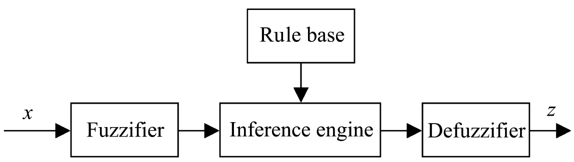


Fig.1 The basic configuration of a fuzzy system

In the fuzzy system in Fig.1, the fuzzifier performs a mapping of the variables $x = (x_1, x_2, \cdots, x_n)^T$ from the crisp input domain $U \subset R^n$ to the fuzzy domain defined in U characterised by membership function $\mu_F : U \rightarrow [0, 1]$, and labelled with linguistic language, such as "Large", "Medium" and "Small". The most commonly used fuzzifier is a "singleton fuzzifier".

A Fuzzy Rule Base consists of a set of linguistic rules in the form of "IF a set of conditions are satisfied, THEN a set of consequences are inferred."

For a fuzzy rule base with N rules, we have:

$$\begin{aligned} &R_j \ : \ If \ x_1 \ is \ A_1^j \ and \ x_2 \ is A_2^j \\ &and \cdots and \ x_n \ is \ A_n^j, \ then \ z \ is \ B^j \end{aligned} \tag{10}$$

where $j = 1, 2, \cdots, N$ (N is the number of fuzzy rules), z is the output of the fuzzy system, and A_i^j and B^j are linguistic terms characterised by fuzzy membership functions $\mu_{A_i^j}(x_i)$ and $\mu_{B^j}(z)$, respectively. Each R_j can be viewed as a fuzzy implication $A_1^j \times \cdots \times A_n^j \rightarrow B^J$, which is a fuzzy set in $U \times R$ with $\mu_{A_1^j \times \cdots \times A_n^j \rightarrow B^j}(x, z) = \mu_{A_1^j}(x_1) \otimes \cdots \otimes \mu_{A_n^j}(x_n) \otimes \mu_{B^j}(z)$. $\otimes$ is a t-norm operation. Commonly used t-norm operations are "product" and "min".

The Fuzzy Inference Engine is decision-making logic which employs fuzzy rules from the fuzzy rule base, to determine a mapping from the fuzzy sets in the input space U to the fuzzy set output space R.

Let A_x be an arbitrary fuzzy set in U, then each R_j determines a fuzzy set $A_x \circ R_j$ in R based on the sup-star composition:

$$\begin{aligned} \mu_{A_x \circ R_j}(z) &= \sup_{x \in U} [\mu_{A_x}(x) \otimes \mu_{A_1^j \times \cdots \times A_n^j \rightarrow B^j}(x, z)] \\ &= \sup_{x \in U} [\mu_{A_x}(x) \otimes \mu_{A_1^j}(x_1) \otimes \cdots \otimes \mu_{A_n^j}(x_n) \otimes \mu_{B^j}(z)] \end{aligned} \tag{11}$$

The Defuzzifier performs a mapping from fuzzy to crisp domain. There are many defuzzification techniques, such as max criterion (MC), mean of maximum (MM) and centre of gravity (COG).

If COG is chosen, the crisp output of the system can be obtained with:

$$z = \frac{\sum_{j=1}^{N} \mu_{A_x \circ R_j}(w_j) w_j}{\sum_{j=1}^{N} \mu_{A_x \circ R_j}(w_j)} \tag{12}$$

where w_j is the point in R at which $\mu_{B^j}(z)$ achieves its maximum value (it is assumed that $\mu_{B^j} = 1$).

The number of fuzzy sets, defined in the input and output universes of discourse, and the number of fuzzy rules in the fuzzy rule base heavily influence the complexity of a fuzzy system, where complexity includes computational complexity, i.e. the computational requirements of the fuzzy system, and space complexity, i.e. the storage requirements of the fuzzy system. These parameters can be viewed as fuzzy system structure parameters. In general, the larger these parameters, the more complex the fuzzy system, and the higher the expected performance of the fuzzy system. Hence, there is always a trade off between complexity and accuracy in the choice of these parameters; and their choice is usually quite subjective.

The linguistic statements of fuzzy rules are the heart of a fuzzy system in the sense that it is these linguistic statements that contain most of the information concerning the fuzzy system design; all other design parameters assist in the effective representation and use of this information. The fuzzy rules usually come from two sources: human experts and training data.

2.4 Fuzzy universal approximation

The set of the fuzzy system described above with a singleton fuzzifier, product inference, and Gaussian membership function consists of functions of the following form:

$$z(x) = \frac{\sum_{j=1}^{N}\Big(\prod_{i=1}^{n}\mu_{A_i^j}(x_i)\Big)\theta_j}{\sum_{j=1}^{N}\Big(\prod_{i=1}^{n}\mu_{A_i^j}(x_i)\Big)} \tag{13}$$

The Gaussian membership function $\mu_{A_i^j}(x_i)$ is defined by:

$$\mu_{A_i^j}(x_i) = \exp\Big[-\Big(\frac{x_i-\zeta_i^j}{\sigma_j}\Big)^2\Big] \tag{14}$$

where ζ_i^j, σ_j are real-valued parameters, and θ_j is the point in R at which $\mu_{B^j}(y)$ achieves its maximum value.

Taking $\dfrac{\Big(\prod_{i=1}^{n}\mu_{A_i^j}(x_i)\Big)}{\sum_{j=1}^{N}\Big(\prod_{i=1}^{n}\mu_{A_i^j}(x_i)\Big)}$ as basis functions and θ_j as constants, $z(x)$ in (13) can be viewed as a linear combination of the basis functions.

Fuzzy basis functions (FBFs) are defined as follows:

$$g_j(x) = \frac{\Big(\prod_{i=1}^{n}\mu_{A_i^j}(x_i)\Big)}{\sum_{j=1}^{N}\Big(\prod_{i=1}^{n}\mu_{A_i^j}(x_i)\Big)}, \quad j = 1, \cdots, N \tag{15}$$

where $\mu_{A_i^j}(x_i)$ are Gaussian membership functions as described in (14). This means that the fuzzy system in (13) is equivalent to an FBF expression:

$$z(x) = \sum_{j=1}^{N} g_j(x)\theta_j = \xi^T\theta \tag{16}$$

where:

$$\xi = [\xi_1, \xi_2, \cdots, \xi_N]^T = [g_1(x), g_2(x), \cdots, g_N(x)]^T$$

and

$$\theta = [\theta_1, \theta_2, \cdots, \theta_N]^T$$

Theorem 1[19]. Suppose $h(x)$ is a continuous function on a compact set U, then for any $\varepsilon \geqslant 0$, there exists a fuzzy logic system like (16), which satisfies:

$$\sup_{x\in U}|h(x) - z(x)| \leqslant \varepsilon \tag{17}$$

Theorem 1 states that the FBF expansions of (13) are universal approximators, and that the theorem is called the fuzzy universal approximation (FUA) theorem.

The following definitions are made for the rest of the paper:

D1. A is a $m \times n$ matrix, a_{ij} is an element of A, that is, $A = \{a_{ij}\}_{m\times n}(i, j = 1, \cdots, m, n)$.

D2. A and B are two $m \times n$ matrices, a_{ij} and b_{ij} are elements of A and B, respectively, $|A| < |B|$ means $|a_{ij}| < |b_{ij}|(i, j = 1, \cdots, m, n)$.

D3. A is a n dimension vector and a_i is an element of A, sgn(A) is the sign of the vector, which means sgn$(A) = \{|a_i|\}_n$.

D4. A and B are n dimension vectors respectively, Pr $oj(A, B)$ is the projection operation of A on B.

The lemmas L1 and L2 together with Barbalat's Lemma are introduced here:

L1. If A and B are two n dimension vectors, then $A^TB = B^TA$.

Proof. As A and B are two n dimension vectors, $A = \{a_i\}_{1\times n}$ and $B = \{b_i\}_{1\times n}$,

$$A^TB = \sum_{i=1}^{n} a_ib_i = \sum_{i=1}^{n} b_ia_i = B^TA$$

□

L2. If A and B are two $m \times n$ matrices, then $(A^TB)^T = B^TA$.

(The proof is similar to that of L1 and omitted).

Barbalat's lemma[17]**.** If the differential function $f(t)$ has a finite limit as $t \to \infty$, and if $\dot{f}(t)$ is uniformly continuous, then $\dot{f}(t) \to 0$ as $t \to \infty$.

3 Controller design

If the system function $F(x)$ and the control gain $G(x)$ are known and there is no disturbance $(d = 0)$, it would be easy to calculate the control law from (9), but in practical applications, $F(x)$ and $G(x)$ are usually not known, and there is external disturbance $(d \neq 0)$, so we need to develop a procedure to design a control law to force system outputs to track reference trajectories with a desired accuracy. In this section, two cases are discussed separately for the designation of control laws of systems in the form of (2) with the integration of FLC, adaptive control, and SMC.

3.1 Case one

A4. In the control system in (2), the control gain matrix is known and it is a unit matrix, that is:

$$G(x) = diag[1, \cdots, 1] \tag{18}$$

As the system function $F(x)$ is not known, in order to design the control law in (9), the fuzzy system $\hat{F}(x|\theta_f)$ is used as an approximation, allowing the control law in (9) to be written in approximated form as:

$$u = -\lfloor R + \hat{F}(x|\theta_f) + d - y' + H(S)\rfloor \tag{19}$$

and

$$\hat{F}(x|\theta_f) = [\hat{f}_1(x|\theta_{f_1}), \cdots, \hat{f}_p(x|\theta_{f_p})]^T = \xi_f^T(x)\theta_f \tag{20}$$

where:

$$\xi_f(x) = diag\lfloor \xi_{f_1}^T(x), \cdots, \xi_{f_p}^T(x)\rfloor$$
$$\theta_f = [\theta_{f_1}^T, \cdots, \theta_{f_p}^T]^T \in R^n \quad (n = \sum_{i=1}^{p} n_i)$$

that is:

$$\hat{f}_i(x|\theta_{f_i}) = \xi_{f_i}^T(x)\theta_{f_i} \quad (i = 1, \cdots, p) \tag{21}$$

where $\xi_{f_i}(x) = [\xi_{f_i,1}(x), \cdots, \xi_{f_i,n_i}(x)]^T \in R^{n_i}$ are FBFs defined as $\xi_{f_i,l}(x) = \dfrac{\prod_{j=1}^{N_i}\mu_{F_j^l}(x_j)}{\sum_{l=1}^{n_i}\prod_{j=1}^{N_i}\mu_{F_j^l}(x_j)}$ $(l = 1, \cdots, n_i$ and N_i is the number of rules for f_i approximation). $\theta_{f_i} \in R^{n_i}$ is an adjustable vector while the membership functions $\mu_{F_j^i}(x_j)$ for $1 \leqslant l \leqslant n_i$ and $1 \leqslant j \leqslant N_i$ are specified beforehand using the knowledge of experts.

Theorem 2. In the control system in (2), if assumption A1, A2 and A4 are satisfied and the control law vector is designed using (19) and (20), and the parameter vectors θ_f are adjusted by the following adaptation law:

$$\mathrm{Pro}j\Big(\xi_f(x)S - \frac{1}{\gamma_f}\dot{\theta}_f, \varphi_f\Big) = 0 \tag{22}$$

γ_f is the adaptation rate, a positive constant. Then closed loop system signals will be bounded and the tracking error vector defined in (2) will be convergent to zero asymptotically.

Proof. If the control is designed by (19), taking it into (7) gives:

$$\begin{aligned}\dot{S} &= R + F(x) + u + d - y'\\ &= R + F(x) - [R + \hat{F}(x|\theta_f) - y' + H(S)] + d - y'\\ &= F(x) - \hat{F}(x|\theta_f) - H(S) + d\end{aligned} \tag{23}$$

Defining the following parameter vector:

$$\theta_f^* = \arg\min_{\theta_f\in\Omega_f}\Big[\sup_{x\in R^n}|\hat{F}(x|\theta_f) - F(x)|\Big]$$

where Ω_f is the constraint set for θ_f and $\omega = F(x) - \hat{F}(x|\theta_f^*)$, then equation (23) can be rewritten as:

$$\begin{aligned}\dot{S} &= F(x) - \hat{F}(x|\theta_f^*) + \hat{F}(x|\theta_f^*) - \hat{F}(x|\theta_f) - H(S) + d\\ &= \omega + [\hat{F}(x|\theta_f^*) - \hat{F}(x|\theta_f)] - H(S) + d\\ &= \omega + \xi_f^T(x)(\theta_f^* - \theta_f) - H(S) + d\\ &= \omega + \xi_f^T(x)\varphi_f - H(S) + d\end{aligned} \tag{24}$$

where $\varphi_f = (\theta_f^* - \theta_f)$.

If the Lyapunov function candidate to be considered is as follows:

$$V = \frac{1}{2}\Big(S^TS + \frac{1}{\gamma_f}\varphi_f^T\varphi_f\Big) \tag{25}$$

and the time derivative of (25) is:

$$\begin{aligned}\dot{V} &= S^T\dot{S} + \frac{1}{\gamma_f}\varphi_f^T\dot{\varphi}_f\\ &= S^T(\omega + \xi_f^T(x)\varphi_f - H(S) + d) + \frac{1}{\gamma_f}\varphi_f^T\dot{\varphi}_f\\ &= S^T(\omega - H(S) + d) + (S^T\xi_f^T(x)\varphi_f + \frac{1}{\gamma_f}\varphi_f^T\dot{\varphi}_f)\end{aligned} \tag{26}$$

let:

$$S^T\xi_f^T(x)\varphi_f + \frac{1}{\gamma_f}\varphi_f^T\dot{\varphi}_f = 0 \tag{27}$$

and $H(S)$ be designed as:

$$H(S) > d_{Max} \tag{28}$$

where:

$$d_{Max} = [d_{1Max}, \cdots, d_{pMax}]^T$$

and d_{iMax}s are the maximum of the absolute values of external disturbances $(i = 1, \cdots, p)$, then equation (26) can be written as:

$$\begin{aligned}\dot{V} &\leqslant S^T\omega - \{|S|^T[H(S) - d]\}\\ &< S^T\omega - \{|S|^T[H(S) - d_{Max}]\}\end{aligned} \tag{29}$$

According to the fuzzy universal approximation theorem, ω is expected to be as small as possible, so the derivative of a Lyapunov candidate is less than zero, that is:

$$\dot{V} < 0 \tag{30}$$

Therefore, all signals in the system are bounded. Since $S(t)$ is uniformly bounded, if $E(0)$ is bounded, then $E(t)$ is also bounded, and since y_r is bounded by design, so $x(t) \in L_\infty$.

To complete the proof and establish asymptotic convergence of the tracking error, that is, $E(t) \to 0$ as $t \to \infty$, it is necessary to show that $S(t) \to 0$ as $t \to \infty$. Applying Barbalat's lemma to:

$$V_1 = V(t) - \int_0^t(\dot{V}(\tau) - S^T(\tau)\omega +$$

$$S^T(\tau)(H(S)-d_{Max}))d\tau \tag{31}$$

$$\text{with}: \ V_1 = S^T\omega - \{|S(\tau)|^T(H(S)-d_{Max})\}$$

it can be easily shown that every term in (25) is bounded, hence $S(t)$ is bounded, which implies that $\dot{V}_1$ is a uniformly continuous function of time. Since V_1 is bounded below by 0, and $\dot{V}_1(t) \leqslant 0$ for all $t \geqslant 0$, from Babarlat's lemma, it can be concluded that $E(t) \to 0$ as $t \to \infty$.

According to L1, $\varphi_f^T\dot{\varphi} = \dot{\varphi}_f^T\varphi$, and as $\dot{\varphi}_f = -\dot{\theta}_f$, then equation (27) can be rewritten as:

$$\mathrm{Pro}j\Big(\xi_f(x)S - \frac{1}{\gamma_f}\dot{\theta}_f, \varphi_f\Big) = 0 \tag{32}$$

3.2 Case two

In the general form of a MIMO non-linear system in (2), it is assumed that assumptions A1–A3 are satisfied. In order to design the control law in (9) the fuzzy system $\hat{F}(x|\theta_f)$ is used to approximate the system function $F(x)$, and the fuzzy system $\hat{G}(x|\theta_G)$ to approximate $G(x)$, the control law in (9) can therefore be written in approximated form as:

$$u = -[\hat{G}(x|\theta_G)]^{-1}[R + \hat{F}(x|\theta_f) + d - y' + H(S)] \tag{33}$$

and:

$$\begin{aligned}\hat{F}(x|\theta_f) &= [\hat{F}_1(x|\theta_{f_1}), \cdots, \hat{f}_p(x|\theta_{f_p})]^T \\ &= \xi_f^T(x)\theta_f\end{aligned} \tag{34}$$

where:

$$\xi_f(x) = diag\lfloor \xi_{f_1}^T(x), \cdots, \xi_{f_p}^T(x)\rfloor$$

$$\theta_f = [\theta_{f_1}^T, \cdots, \theta_{f_p}^T]^T \in R^n \quad (n = \sum_{i=1}^{p} n_i)$$

that is, $\hat{f}_i(x|\theta_{f_i}) = \xi_{f_i}^T(x)\theta_{f_i}(i = 1, \cdots, p)$ where $\xi_{f_i}(x) = [\xi_{f_i,1}(x), \cdots, \xi_{f_i,n_i}(x)]^T \in R^{n_i}$ are FBFs defined as $\xi_{f_i,l}(x) = \dfrac{\prod_{j=1}^{N_i}\mu_{F_j^l}(x_j)}{\sum_{l=1}^{n_i}\prod_{j=1}^{N_i}\mu_{F_j^l}(x_j)}(l = 1, \cdots, n_i,\ N_i$ is the number of the rules for f_i approximation), $\theta_{f_i} \in R^{n_i}$ is an adjustable vector, the membership functions $\mu_{F_j^i}(x_j)$ for $1 \leqslant l \leqslant m_{f_i}$ and $1 \leqslant j \leqslant k_i$ are specified beforehand using the knowledge of experts.

Also:

$$\begin{aligned}\hat{G}(x|\theta_G) &= [\hat{G}_1(x|\theta_G), \cdots, \hat{G}_p(x|\theta_G)]^T \\ &= \xi_G^T(x)\theta_G\end{aligned} \tag{35}$$

where:

$$\xi_G(x) = diag\lfloor \xi_{G_1}^T(x), \cdots, \xi_{G_p}^T(x)\rfloor$$

$$\theta_G = [\theta_{G_1}^T, \cdots, \theta_{G_p}^T]^T \in R^n \quad (n = \sum_{j=1}^{p} n_i)$$

that is:

$$\hat{G}_i(x|\theta_G) = \xi_{G_i}^T(x)\theta_{G_i} \quad (i = 1, \cdots, p) \tag{36}$$

where $\xi_{G_i}(x) = [\xi_{G_i,1}(x), \cdots, \xi_{G_i,n_i}(x)]^T \in R^{n_i}$ are FBFs defined as $\xi_{G_i,l}(x) = \dfrac{\prod_{j=1}^{N_g}\mu_{G_j^l}(x_j)}{\sum_{l=1}^{n_i}\prod_{j=1}^{N_g}\mu_{G_j^l}(x_j)}(l = 1, \cdots, n_i,\ N_g$ is the number of fuzzy rules) and $\theta_{G_i} \in R^{n_i \times n_i}$ is an adjustable matrix and the membership functions $\mu_{G_j^i}(x_j)$ for $1 \leqslant l \leqslant m_{G_i}$ and $1 \leqslant j \leqslant N_g$ are specified beforehand using the knowledge of experts.

Theorem 3. For the control system in (2), if the assumption A1–A3 are satisfied and the control law vector is designed using (33), (34) and (35), and parameter vectors θ_f and θ_G are adjusted by the following adaptation laws:

$$\mathrm{Rro}j\Big(\xi_f(x)S - \frac{1}{\gamma_f}\dot{\theta_f}, \varphi_f\Big) = 0 \tag{37}$$

and:

$$\mathrm{Pro}j\Big(\xi_G(x)S - \frac{1}{\gamma_G}\dot{\theta_G}, \varphi_G\Big) = 0 \tag{38}$$

γ_f and γ_G are the adaptation rates respectively, positive constants. Then closed loop system signals will be bounded and the tracking error vector defined in (2) will be convergent to zero asymptotically.

Proof. The following equation can be obtained from (7) as:

$$\begin{aligned}\dot{S} &= R + F(x) + G(x)u + d - y' \\ &= F(x) - \hat{F}(x|\theta_f^*) + [\hat{F}(x|\theta_f^*) - \hat{F}(x|\theta_f)] \\ &\quad + \{[G(x) - \hat{G}(x|\theta_G^*)] + [\hat{G}(x|\theta_G^*) - \hat{G}(x|\theta_G)]\}u \\ &\quad - H(S) - y' \\ &= F(x) - \hat{F}(x|\theta_f^*) + [G(x) - \hat{G}(x|\theta_G^*)] \\ &\quad + \hat{F}(x|\theta_f^*) - \hat{F}(x|\theta_f)] + [\hat{G}(x|\theta_G^*) - \hat{G}(x|\theta_G)]u \\ &\quad - H(S) - y'\end{aligned} \tag{39}$$

By defining the following parameter vectors:

$$\theta_f^* = \arg\min_{\theta_f \in \Omega_f}\Big[\sup_{x \in R^n}|\hat{F}(x|\theta_f) - F(x)|\Big]$$

$$\theta_G^* = \arg\min_{\theta_G \in \Omega_G}\Big[\sup_{x \in R^n}|\hat{G}(x|\theta_G) - G(x)|\Big]$$

where Ω_f and Ω_G are constraint sets for θ_f and θ_G, respectively, and $\omega = F(x)-\hat{F}(x|\theta_f^*)+(G(x)-\hat{G}(x|\theta_G^*))u$. Equation (39) can be rewritten as:

$$\begin{aligned}\dot{S} &= F(x) - \hat{F}(x|\theta_f^*) + \hat{F}(x|\theta_f^*) - \hat{F}(x|\theta_f) + [G(x) \\ &\quad - \hat{G}(x|\theta_G^*)]u + [\hat{G}(x|\theta_G^*) - \hat{G}(x|\theta_G)]u - H(S) + d \\ &= \omega + [\hat{F}(x|\theta_f^*) - \hat{F}(x|\theta_f)] - [\hat{G}(x|\theta_G^*) - \hat{G}(x|\theta_G)]u \\ &\quad - y' - H(S) + d \\ &= \omega + \xi_f^T(x)(\theta_f^* - \theta_f) + \xi_G^T(x)(\theta_G^* - \theta_G)u - H(S) + d \\ &= \omega + \xi_f^T(x)\varphi_f + \xi_G^T(x)\varphi_G u - H(S) + d \qquad (40)\end{aligned}$$

where $\varphi_f = (\theta_f^* - \theta_f)$ and $\varphi_G = (\theta_G^* - \theta_G)$.

The Lyapunov function candidate to be considered is as follows:

$$V = \frac{1}{2}\Big(S^TS + \frac{1}{\gamma_f}\varphi_f^T\varphi_f + \frac{1}{\gamma_G}\varphi_G^T\varphi_G\Big) \qquad (41)$$

The time derivative of (41) is therefore:

$$\begin{aligned}\dot{V} &= S^T\dot{S} + \frac{1}{\gamma_f}\varphi_f^T\dot{\varphi}_f + \frac{1}{\gamma_G}\varphi_G^T\dot{\varphi}_G \\ &= S^T(\omega + \xi_f^T(x)\varphi_f + \xi_G^T(x)\varphi_G u - H(S) + d) \\ &\quad + \frac{1}{\gamma_f}\varphi_f^T\dot{\varphi}_f + \frac{1}{\gamma_G}\varphi_G^T\dot{\varphi}_G \\ &= S^T(\omega - H(S) + d) + \Big(S^T\xi_f^T(x)\varphi_f + \frac{1}{\gamma_f}\varphi_f^T\dot{\varphi}_f\Big) \\ &\quad + \Big(S^T\xi_G^T(x)\varphi_G u + \frac{1}{\gamma_G}\varphi_G^T\dot{\varphi}_G\Big) \qquad (42)\end{aligned}$$

Let

$$S^T\xi_f^T(x)\varphi_f + \frac{1}{\gamma_f}\varphi_f^T\dot{\varphi}_f = 0 \qquad (43)$$

and

$$S^T\xi_G^T(x)\varphi_G u + \frac{1}{\gamma_G}\varphi_G^T\dot{\varphi}_G = 0 \qquad (44)$$

and $H(S)$ is designed as in Case One, then equation (42) can be written as:

$$\begin{aligned}\dot{V} &\leqslant S^T\omega - \{|S|^T[H(S) - d]\} \\ &< S^T\omega - \{|S|^T[H(S) - d_{Max}]\}\end{aligned}$$

The completion of the proof is similar to that of the proof for Theorem 1.

From (43) and (44), equations (37) and (38) can be obtained.

3.3 Design procedure

From the discussion above, a general controller design procedure can be summarised as follows:

1) Specify the sliding function vector $S = [s_1, \cdots, s_p]^T$ in (3) and identify the tracking error in (4);

2) Select the Hurwitzian coefficient vectors C_i to guarantee that all roots of the sliding function vector $S = [s_1, \cdots, s_p]^T$ in (3) are on the left side of the plane,

3) Design the membership functions of the fuzzy sets for variables $x_1, \cdots, x_p$ with $\mu_i^j(x_i)(i = 1, \cdots, p)$ and $j = 1, \cdots, m_i$, where m_i is the number of fuzzy sets for variable x_i;

4) Construct the fuzzy rule base for approximating $F(x)$ and $G(x)$,

5) Specify the adaptive rates r_f and r_G;

6) Calculate the approximate $\hat{F}(x|\theta_f)$ and $\hat{G}(x|\theta_G)$ in (34) and (35);

7) Adjust the parameter vectors θ_f and θ_G with the adaptation laws (37) and (38), respectively.

In the next section, two examples are studied to verify the effectiveness of the proposed fuzzy adaptive sliding mode control strategy.

4 Simulation studies

In this section two examples are studied for the different cases discussed in Section 3.

4.1 Example 1

Consider the following simple non-linear MIMO system[29]:

$$\begin{aligned}\dot{x}_1 &= x_2 + u_1 \qquad (45)\\ \dot{x}_2 &= x_1 + (2e^{-(x_1^2+x_2^2)} - 0.1)x_2 + u_2\end{aligned}$$

The systems outputs are:

$$y_1 = x_1, \qquad y_2 = x_2$$

The system is unstable if it is free of control.

The system dynamics can be written in a matrix as:

$$\begin{aligned}\dot{x} &= Ax + B[F(x) + G(x)u + d] \qquad (46)\\ y &= Cx\end{aligned}$$

where:

$x = [x_1 \quad x_2]^T \in R^2$ is the system state vector.

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$F(x) = \begin{bmatrix} x_2 \\ x_1 + (2e^{-(x_1^2+x_2^2)} - 0.1)x_2 \end{bmatrix}$$

$$G(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$ (is an external disturbance vector)

It is assumed that assumption A1 is satisfied, that the system function vector $F(x)$ is not exactly known, but it is bounded and smooth, meeting the requirement of A2, and that the system control gain matrix $G(x)$

satisfies assumption A4. The objective is to design the control law vector $u = [u_1 \quad u_2]^T$ to force the state vector of the system to track the reference trajectory vector $y_r = [y_{r1} \quad y_{r2}]^T$ where $y_{r1} = \begin{cases} 0.5, & 0 < t \leqslant 0.5 \\ -0.5 & 0.5 < t \leqslant 1 \end{cases}$ and $y_{r2} = 0.5\sin(2\pi t)$ with the function vector $F(x)$ unknown.

The sliding function vector is designed as:

$$S = [s_1 \quad s_2]^T \tag{47}$$

where $s_1 = e_1 = x_1 - y_{r1}$ and $s_2 = e_2 = x_2 - y_{r2}$.

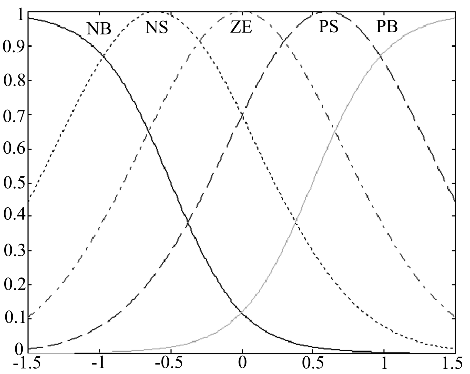
Because the system function vector is not known exactly, an FUA theorem is applied to approximate it.

The membership functions of the fuzzy sets for variables x_1 and x_2 in $(x_1, x_2) \in ([-1.5, 1.5] \times [-1.5, 1.5])$ are selected as:

$$\begin{aligned}
\mu_i^1(x_i) &= \frac{1}{1+\exp(4\times(x_i+0.5))} \text{ for Negative Big (NB)} \\
\mu_i^2(x_i) &= \exp(-(x_i+0.6)^2) \text{ for Negative Small (NS)} \\
\mu_i^3(x_i) &= \exp(-(x_i)^2) \text{ for ZEro (ZE)} \\
\mu_i^4(x_i) &= \exp(-(x_i-0.6)^2) \text{ for Positive Small (PS)} \\
\mu_i^5(x_i) &= \frac{1}{1+\exp(-4\times(x_i-0.5))} \text{for Positive Big (PB)}
\end{aligned} \tag{48}$$

where $i = 1, 2$.

The membership functions are shown in Fig.2.



solid: NB, dotted: NS, dash dotted: ZE, dashed: PS, solid: PB

Fig.2 Membership functions for the state variables x_1 and x_2

The fuzzy rules are constructed as follows:

$$\begin{aligned}
&R_l : IF\ x_1\ is\ A_1^l\ and\ x_2\ is\ A_2^l \\
&\qquad THEN\ \hat{F}^l\ is\ B^l \quad (l = 1, \cdots, 25)
\end{aligned} \tag{49}$$

where $A_i^l (i = 1, 2)$ and B^l are linguistic terms characterised by the related fuzzy membership functions defined in (10).

In order to simplify computation, $f_1(x)$ is assumed to be dominated only by x_2. This allows the system function vector to be approximated as:

$$\hat{F}(x|\theta_f) = [\hat{f}_1(x|\theta_{f1} \quad \hat{f}_2(x|\theta_{f2})]^T \tag{50}$$

and:

$$\begin{aligned}
\hat{f}_1(x|\theta_{f1}) &= \xi_{f1}^T \theta_{f1} \\
\hat{f}_2(x|\theta_{f2}) &= \xi_{f2}^T \theta_{f2}
\end{aligned} \tag{51}$$

where $\xi_{f1} = [\xi_{f11}, \cdots \xi_{f1i}, \cdots, \xi_{f15}]^T$ with $\xi_{f1i} = \mu_2^i(x_i)/D_{f1}$ and $D_{f1} = \sum_{i=1}^{5} \mu_2^i(x_i)$, and $\xi_{f2} = [\xi_{f211}, \cdots \xi_{f2ij}, \cdots, \xi_{f255}]^T$ with $\xi_{f2ij} = \mu_1^i(x_i)\mu_2^j(x_j)/D_{f_2}$ and $D_{f2} = \sum_{i,j=1}^{5} \mu_1^i(x_i)\mu_2^j(x_j)$.

$H(S)$ is designed as $H(S) = diag[0 \quad 0] sign \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} + diag[20 \quad 20] \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$.

The system's initial condition is set as $x(0) = [0 \quad 0]^T$, $\theta_{f2}(0)$ randomly selected within $[-1 \quad 1]$, sampling rate set to $\triangle T = 0.01$ of a second, simulation period T selected as 2 seconds, and the adaptive rate specified as $r_{f2} = 40$.

Simulation results for the system are shown in Fig.3.

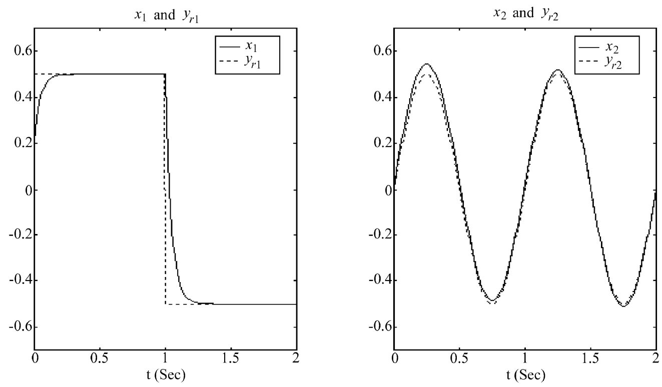


Fig.3 Simulation results for example 1

4.2 Example 2

The second example of a MIMO non-linear system to be considered is as:

$$\begin{aligned}
\dot{x_1} &= x_3 + x_3 x_2 + u_1 + d_1 \\
\dot{x_2} &= x_3 \\
\dot{x_3} &= x_1^2 + x_1 x_3 + x_3^2 + (e^{-(x_1^2+x_2^2+x_3^2)} \\
&\quad + \sin(2\pi x_3))u_1 + 2u_2 + d_3
\end{aligned} \tag{52}$$

where system outputs are:

$$y_1 = x_1, \quad y_2 = x_2$$

The differential equations of the system in (52) can be rewritten in matrix form as:

$$\begin{aligned}\dot{x} &= A + B[F(x) + G(x)u + d] \\ y &= C^T x\end{aligned} \tag{53}$$

where $x = [x_1, x_2, x_3]^T \in R^3$ is the system state vector:

$$\begin{aligned}A &= diag[A_1, A_2] \\ B &= diag[B_1, B_2] \\ C &= diag[C_1, C_2]\end{aligned}$$

$$A_1 = 0, A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B_1 = 1, B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C_1 = 1, C_2 = [1 \quad 0]$$

$$F(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} = \begin{bmatrix} x_3 + x_3 x_2 \\ x_1^2 + x_1 x_3 + x_3^2 \end{bmatrix}$$

$$G(x) = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ e^{-(x_1^2+x_2^2+x_3^2)} + \sin(2\pi x_3) & 2 \end{bmatrix}$$

$$d = \begin{bmatrix} d_1 \\ d_3 \end{bmatrix}$$

and the complete matrix form expression of (52) can be described as:

$$\begin{aligned}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ &+ \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \left\{ \begin{bmatrix} x_3 + x_3 x_2 \\ x_1^2 + x_1 x_3 + x_3^2 \end{bmatrix} \right. \\ &+ \begin{bmatrix} 1 & 0 \\ e^{-(x_1^2+x_2^2+x_3^2)} + \sin(2\pi x_3) & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ &\left. + \begin{bmatrix} d_1 \\ d_3 \end{bmatrix} \right\} \\ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\end{aligned} \tag{54}$$

It is assumed that the system state vector $x = [x_1, x_2, x_3]^T$ is measurable, that means, the system meets assumption A1, and the system function vector $F(x) = [f_1(x), f_2(x)]^T$ is not known exactly, but it is bounded and smooth; the control gain matrix $G(x) = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$ is partly known and is nonsingular ($g_{11} = 1, g_{12} = 0, g_{22} = 2$ and g_{21} is known). The objective of the system design is to determine the output of the fuzzy logic controller $u = [u_1 \quad u_2]^T$ based on an adaptive law to track the desired trajectory $y_r = [y_{r1} \quad y_{r2}]^T = [0.2 \sin \frac{\pi}{2} t \quad 0.2 \cos \frac{\pi}{2} t]^T$ where all signals involved are uniformly bounded and tracking error converges to zero.

In this case, the system satisfies assumptions A1–A3. In order to reduce computational burden, it is assumed that $f_1(x)$ is known to be only associated with x_2 and x_3, $f_2(x)$ only associated with x_1 and x_3, and g_{21} related to x_1, x_2 and x_3. The FUA theorem will be applied to approximate unknown functions.

The sliding function vector can be described as:

$$S = [s_1 \quad s_2]^T \tag{55}$$

where $s_1 = e_1 = x_1 - y_{r1}$ and $s_2 = e_2 = k_{21}(x_2 - y_{r2}) + x_3 - y_{r3}, (y_{r3} = \dot{y}_{r2} = -0.1\pi \sin \frac{\pi}{2} t)$.

Five fuzzy sets are designed for each of the crisp variables x_1, x_2 and x_3 in $(x_1, x_2, x_3) \in ([-1.5, 1.5] \times [-1.5, 1.5] \times [-1.5, 1.5])$, and the membership functions of the fuzzy sets are selected as in Case One.

As $f_1(x)$ is only associated with x_2 and x_3, and $f_2(x)$ is only associated with x_1 and x_3, the following rules can be drawn to approximate $\hat{f}_1(x)$ and $\hat{f}_2(x)$:

R^i : *If* x_2 *is* A_2^i *and* x_3 *is* A_3^i *then* $\hat{f}_1^i$ *is* B_1^i, $i = 1, \cdots, 25$

R^j : *If* x_1 *is* A_1^j *and* x_3 *is* A_3^j *then* $\hat{f}_2^j$ *is* B_2^j, $j = 1, \cdots, 25$

R^k : *If* x_1 *is* A_1^k *and* x_2 *is* A_2^k *and* x_3 *is* A_3^k *then* $\hat{g}_{21}^k$ *is* B_{21}^k, $k = 1, \cdots, 125$ (56)

where A and B are linguistic terms characterised by related fuzzy membership functions, and $f_1(x)$ is approximated by:

$$\hat{f}_1(x|\theta_{f1}) = \xi_{f1}^T \theta_{f1} \tag{57}$$

where $\xi_{f1} = [\xi_{f1}^{11}(x), \cdots, \xi_{f1}^{55}(x)]$ and

$$\xi_{f1}^{ij}(x) = \frac{\mu_2^i(x_2)\mu_3^j(x_3)}{\sum_{i,j=1}^{5} \mu_2^i(x_2)\mu_3^j(x_3)}, \ i, j = 1, \cdots, 5$$

and $f_2(x)$ is approximated by:

$$\hat{f}_2(x|\theta_{f2}) = \xi_{f2}^T \theta_{f2} \tag{58}$$

where $\xi_{f2} = [\xi_{f2}^{11}(x), \cdots, \xi_{f2}^{55}(x)]$ and

$$\xi_{f2}^{ij}(x) = \frac{\mu_1^i(x_1)\mu_3^j(x_3)}{\sum_{i,j=1}^{5} \mu_1^i(x_1)\mu_3^j(x_3)}, \ i, j = 1, \cdots, 5$$

and $g_{21}(x)$ is approximated by:

$$\hat{g}_{21}(x|\theta_{g21}) = \xi_{g21}^T \theta_{g21} \tag{59}$$

where $\xi_{g21} = [\xi_{g21}^{111}(x), \cdots, \xi_{g21}^{555}(x)]^T$ and

$$\xi_{g21}^{ijk}(x) = \frac{\mu_1^i(x_1)\mu_2^j(x_2)\mu_3^k(x_3)}{\sum_{i,j=1}^{5} \mu_1^i(x_1)\mu_2^j(x_2)\mu_3^k(x_3)}, \quad i, j, k = 1, \cdots, 5$$

According to the adaptive laws (37) and (38), adaptive rates are selected as $\gamma_{f1} = 10, \gamma_{f2} = 10$ and $\gamma_{g21} = 10$, and the parameters θ_{f1}, θ_{f2} and θ_{g21} are adjusted by:

$$\begin{aligned} \dot{\theta}_{f1} &= \gamma_{f1} s_1 \xi_{f1} \\ \dot{\theta}_{f2} &= \gamma_{f2} s_2 \xi_{f2} \\ \dot{\theta}_{g21} &= \gamma_{g21} s_2 \xi_{g21} u_1 \end{aligned} \tag{60}$$

and $H(S)$ is designed as $H(S) = diag[0 \quad 0] sign \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} + diag[20 \quad 8] \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$.

The initial conditions of the system are set as $x(0) = [0.1 \quad 0 \quad -0.1]^T$, $\theta_{f1}(0)$, $\theta_{f2}(0)$ and $\theta_{g21}(0)$ are randomly selected within $[-1 \quad 1]$, sampling rate is set to $\triangle T = 0.01$ of a second, and the simulation period is selected as 4 seconds. The simulation results are shown in Fig.4.

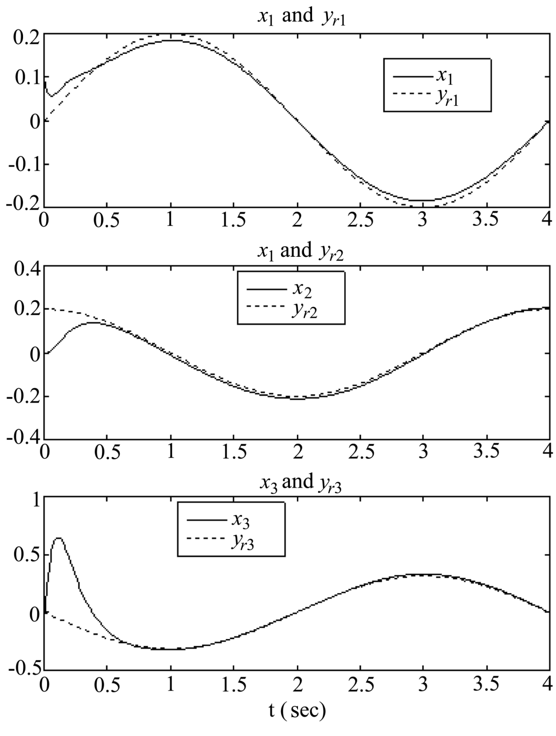


Fig.4 The simulation results for example 2

From the simulation results shown in Figs.3 and 4, we can see that the proposed SMC scheme based on fuzzy adaptive law is very effective for tracking control of MIMO nonlinear systems with uncertainty. The proposed control algorithm overcomes the chatter disadvantage of a pure SMC.

5 Conclusions

A SMC algorithm based on indirect fuzzy adaptive law is proposed for the tracking control problem of the general form of MIMO nonlinear systems with uncertainty. The proposed algorithm takes advantage of SMC, FLC and adaptive control with a reaching law method and fuzzy universal approximation, and does not need to know much about the structure and bounds of the parameters of systems as in the design of conventional SMCs. The stability of the control system is proved in terms of a Lyapunov second stability theorem. Simulation studies in this paper show the effectiveness of the proposed hybrid control algorithm.

References

[1] E. H. Mandani, Applications of fuzzy algorithms for simple dynamic plants, *Proceedings IEE*, vol. 121, pp. 1585–1588, 1974.

[2] L. A. Zadeh, Fuzzy sets, *Information and Control*, vol. 8, pp. 338–353, 1965.

[3] S. V. Emelyanov, *Variable Structure Control Systems*, Moscow: Nauka, 1967(in Russian).

[4] Y. Itkis, *Control Systems of Variable Structure*, New York: Wiley, 1976.

[5] V. I. Utkin, Variable structure systems with sliding modes, *IEEE Transactions on Automatic Control*, vol. 22, no. 2, pp. 212–222, April, 1977.

[6] J. Y. Hung, W. Gao, J. C. Hung, Variable structure control: a survey, *IEEE Transaction on Industrial Electronics*, vol. 40, no. 1, pp. 2–22, February, 1993.

[7] R. A. DeClarlo, S. H. Zak, G. R. Mathews, Variable structure control of nonlinear multivariable systems: A Tutorial, *Proceedings of IEEE*, vol. 76, no. 3, pp. 212–232, 1988.

[8] J. J. E. Slotine, W. P. Li, *Applied Nonlinear Control*, Prentice-Hall, Englewood Cliffs, NJ, 1991.

[9] V. I. Utkin, *Sliding Modes in Control Optimization*, New York, Springer-Verlag, 1992.

[10] A. S. I. Zinober, An introduction to sliding mode variable structure control, *Variable Structure and Lyapunov Control*, A. S. Zinoear (Ed.), London, UK: Springer-Verlag, 1994.

[11] V. I. Utkin, VSS Premise in XX Century: Evidences of a witness, *Proceedings of the 6th. IEEE International Workshop on Variable Structure Systems* (*VSS' 2000*), Gold Coast, Coolangatta, Australia, 7–9, December 2000.

[12] R. Palm, Sliding mode fuzzy control, *Proceedings of IEEE Conference on Fuzzy Systems*, San Diego, pp. 519–526, 1992.

[13] O. Kaynak, K. Erbatur, M. Ertugrul, The fusion of computationally intelligent methodologies and sliding-mode control: a survey, *IEEE Transactions on Industrial Electronics*, vol. 48, no. 1, pp. 4–17, February, 2001.

[14] J. Park, J. Kim, D. Park, LMI-based design of stabilizing fuzzy controllers for nonlinear systems described by Takagi-Sugeno fuzzy model, *Fuzzy Sets and Systems*, vol. 122, pp. 73–82, 2001.

[15] W. Chang, J. B. Park, Y. H. Joo, G. Chen, Design of robust fuzzy-model-based controller with sliding mode control for SISO nonlinear systems, *Fuzzy Sets and Systems*, vol. 125, pp. 1–22, 2002.

[16] F. Qiao, Q. M. Zhu, A. Winfield, C. Melhuish, Fuzzy sliding mode control for discrete time nonlinear systems, *Computing Technology and Automation*, vol. 22, no. 2, pp. 311–316, June, 2003.

[17] K. J. Åström, B. Wittenmark, *Adaptive Control*, 2nd Ed., MA: Addison-Wesley, 1995.

[18] W. M. Haddad, T. Hayakawa, V. Chellaboina, Robust adaptive control for nonlinear uncertain systems, *Automatica*, vol. 39, pp. 551–556, 2003.

[19] L. X. Wang, *Fuzzy Systems and Control*, PTR Prentice Hall, Englewood Cliffs, New Jersey, 1994.

[20] C. Y. Su, Y. Stepanenko, Adaptive control of a class of nonlinear systems with fuzzy logic, *IEEE Transactions on Fuzzy Systems*, vol. 2, no. 4, pp. 285−294, November, 1994.

[21] B, Yoo, W. Ham, Adaptive fuzzy sliding mode control of nonlinear system, *IEEE Transactions on Fuzzy Systems*, vol.6, no. 2, pp. 315−321, May, 1998.

[22] S. C. Tong, Q. G. Li, T. Y. Chai, Fuzzy adaptive control for a class of nonlinear systems, *Fuzzy Sets and Systems*, vol. 101, pp. 31−39, 1999.

[23] T. Y. Chai, S. C. Tong, Fuzzy direct adaptive control for a class of nonlinear systems, *Fuzzy Sets and Systems*, vol. 103, pp. 379−387, 1999.

[24] J. Wang, A. B. Rad, P. T. Chan, Indirect adaptive fuzzy sliding mode control, Part I: Fuzzy switching, *Fuzzy Sets and Systems*, vol. 122, pp. 21−30, 2001.

[25] P. T. Chan, A. B. Rad, J. Wang, Indirect adaptive fuzzy sliding mode control, Part II: Parameter projection and supervisory control, *Fuzzy Sets and Systems*, vol. 122, pp. 31−443, 2001.

[26] Y. C. Chang, Robust tracking control for nonlinear MIMO systems via fuzzy approaches, *Automatica*, vol. 36, pp. 1535−1545, 2000.

[27] H. X. Li, S. C. Tong, A hybrid adaptive fuzzy control for a class of nonlinear MIMO systems, *IEEE Transactions on Fuzzy Systems*, vol. 11, no. 1, pp. 24−34, February, 2003.

[28] S. C. Tong, H. X. Li, Fuzzy adaptive sliding mode control for MIMO nonlinear systems, *IEEE Transactions on Fuzzy Systems*, vol. 11, no. 3, pp. 354−360, June, 2003.

[29] S. Jagannathan, M. W. Vandegrift, F. L. Lewis, Adaptive fuzzy logic control of discrete-time dynamical systems, *Automatica*, vol. 36, pp. 229−241, 2000.

Feng Qiao received the B.S. degree in electrical engineering and M.S. degree in systems engineering from the Northeastern University, Shenyang, China, in 1982 and 1987, respectively. During the period between 1987 and 2001, he worked at the Automation Research Institute of Metallurgical Industry (ARIM), Beijing, China, and he left ARIM on the post of a senior engineer in electrical and computer engineering. Now, he is a research PhD student at the University of the West of England, Bristol, UK, in intelligent modelling and control, his research interests include fuzzy logic systems, neural networks, nonlinear systems, stochastic systems, Kalman filter, sliding mode control, robust control, adaptive control, system identification, mathematical programming and optimisation, software development.

Quanmin Zhu had higher education both in China and the UK. He obtained his PhD in Faculty of Engineering, University of Warwick, UK in 1989. Currently Dr Zhu is a reader in Electronics, Faculty of Computing, Engineering and Mathematical Sciences (CEMS) University of the West of England (UWE), Bristol, UK. His main research interest is in the area of nonlinear system modelling, identification, and control. Recently Dr Zhu started investigating electrodynamics of acupuncture points and sensory stimulation effects in human body, modelling of human meridian systems, and building up electro-acupuncture instruments. He has published over ninety papers on these topics.

Allen FT Winfield in 1984, shortly after completing a PhD in Digital Communications, Alan Winfield gave up his lectureship at the University of Hull to found a company on the newly established Hull Science Park. Dr Winfield went on to establish APD Communications Ltd as one of the key UK providers of software for safety-critical mobile radio systems. He left APD in 1991 to take up appointment as Associate Dean (Research) and Hewlett-Packard Professor of Electronic Engineering at the University of the West of England. Moving into the field of mobile robotics, he co-founded the Intelligent Autonomous Systems Laboratory in 1993. His work is centred on Control and Communications architectures for mobile robots. Current research has three strands: ad-hoc wireless connected robot swarms; autonomy in space robotics, and provably-stable intelligent control.

Chris Melhuish is Professor and Director of the Intelligent Autonomous Systems Laboratory of the University of the West of England (UWE). He has degrees in Geology from Durham University, an MSc in Computer Science from Bristol University and a PhD in collective robotics from UWE. He is a member of the British Computer Society and is a chartered engineer. His research interests include mobile robotics and in particular minimalist collective robotics, aerial robot formation control and robot energy autonomy.